HW #2

1. 1D scattering

(a)

The delta function potential requires a discontinuity of the wave function, $\psi'(+0) - \psi'(-0) = \frac{2 m \mu}{\hbar^2} \psi(0)$, while the wavefunction itself is continuous.

$$\text{Solve}\left[\left\{1 + R == T,\ I\,k\,T - I\,k\,(1 - R) == \frac{2\,m\,\mu}{\hbar^2}\,T\right\},\ \{R, T\}\right]$$

$$\left\{\left\{R \to -\frac{i\,m\,\mu}{i\,m\,\mu + k\,\hbar^2},\ T \to \frac{k\,\hbar^2}{i\,m\,\mu + k\,\hbar^2}\right\}\right\}$$

$$\text{Simplify}[\%]$$

$$\left\{\left\{R \to \frac{m\,\mu}{-m\,\mu + i\,k\,\hbar^2},\ T \to \frac{k\,\hbar^2}{i\,m\,\mu + k\,\hbar^2}\right\}\right\}$$

$$\text{Simplify}\left[\left(-\frac{m\,\mu}{m\,\mu - i\,k\,\hbar^2}\right)\left(-\frac{m\,\mu}{m\,\mu + i\,k\,\hbar^2}\right) + \left(\frac{k\,\hbar^2}{i\,m\,\mu + k\,\hbar^2}\right)\left(\frac{k\,\hbar^2}{-i\,m\,\mu + k\,\hbar^2}\right)\right]$$

$$1$$

It is worth noting that the reflection probabiliy $|R|^2$ is 100% at the low momentum limit $k = 0$, while it monotonically decreases to zero at higher momenta.

(b)

We assume T and R to be approximately constant. Hence the Gaussian wave packet is given by the integrals

$$\text{Integrate}\left[E^{I\,q\,x}\,E^{-(q-k)^2/2/\sigma^2}\,E^{-I\,(\hbar^2\,q^2/2/m)\,t/\hbar},\ \{q, -\infty, \infty\},\ \text{Assumptions} \to \{\sigma \in \text{Reals}\ \&\&\ m \in \text{Reals}\ \&\&\ \hbar \in \text{Reals}\ \&\&\ t \in \text{Reals}\}\right]$$

$$\frac{e^{\frac{2\,i\,k\,m\,x - m\,x^2\,\sigma^2 - i\,k^2\,t\,\hbar}{2\,m + 2\,i\,t\,\sigma^2\,\hbar}}\,\sqrt{2\,\pi}}{\sqrt{\frac{1}{\sigma^2} + \frac{i\,t\,\hbar}{m}}}$$

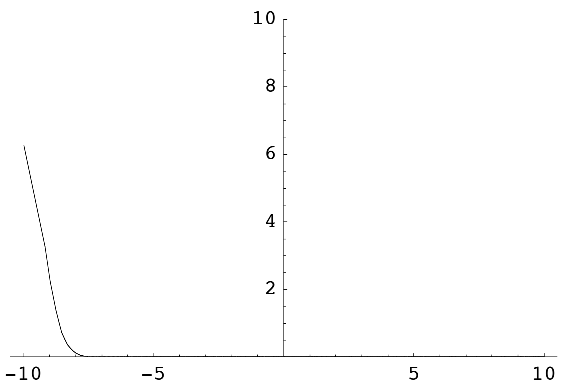
$$\text{Integrate}\left[E^{-I\,q\,x}\,E^{-(q-k)^2/2/\sigma^2}\,E^{-I\,(\hbar^2\,q^2/2/m)\,t/\hbar},\ \{q, -\infty, \infty\},\ \text{Assumptions} \to \{\sigma \in \text{Reals}\ \&\&\ m \in \text{Reals}\ \&\&\ \hbar \in \text{Reals}\ \&\&\ t \in \text{Reals}\}\right]$$

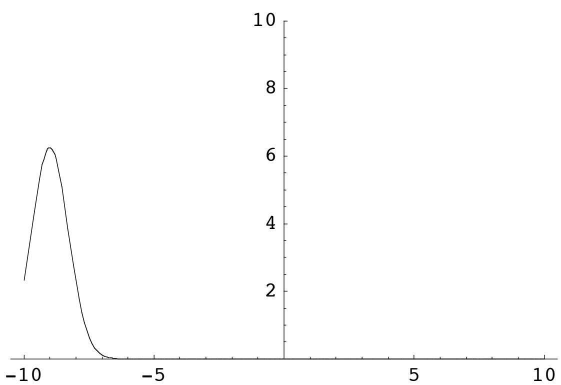
$$\frac{e^{\frac{2\,i\,k\,m\,x + m\,x^2\,\sigma^2 + i\,k^2\,t\,\hbar}{-2\,m - 2\,i\,t\,\sigma^2\,\hbar}}\,\sqrt{2\,\pi}}{\sqrt{\frac{1}{\sigma^2} + \frac{i\,t\,\hbar}{m}}}$$

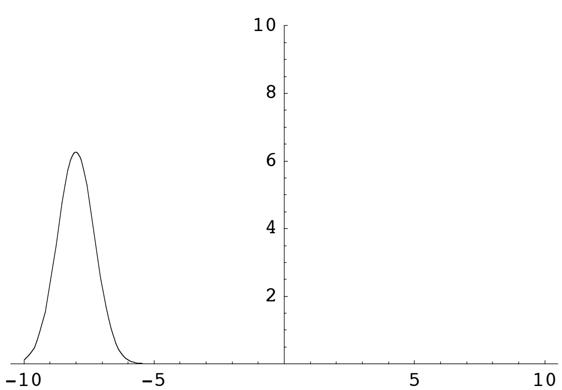
$$\psi = \text{If}\left[x < 0,\ \frac{e^{\frac{2 i k m x - m x^2 \sigma^2 - i k^2 t \hbar}{2 m + 2 i t \sigma^2 \hbar}} \sqrt{2\pi}}{\sqrt{\frac{1}{\sigma^2} + \frac{i t \hbar}{m}}} + R \frac{e^{-\frac{2 i k m x + m x^2 \sigma^2 + i k^2 t \hbar}{2 m + 2 i t \sigma^2 \hbar}} \sqrt{2\pi}}{\sqrt{\frac{1}{\sigma^2} + \frac{i t \hbar}{m}}},\ T \frac{e^{\frac{2 i k m x - m x^2 \sigma^2 - i k^2 t \hbar}{2 m + 2 i t \sigma^2 \hbar}} \sqrt{2\pi}}{\sqrt{\frac{1}{\sigma^2} + \frac{i t \hbar}{m}}}\right]$$

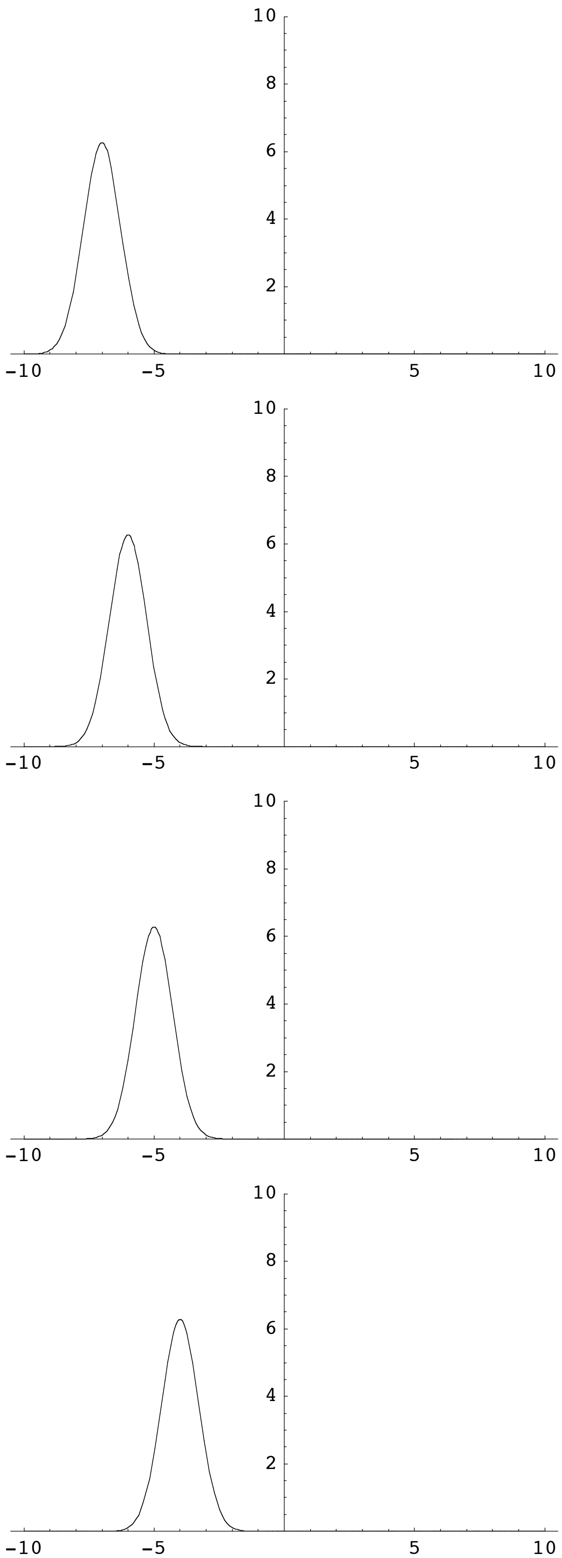
$$\text{If}\left[x < 0,\ \frac{e^{\frac{2 i k m x - m x^2 \sigma^2 - i k^2 t \hbar}{2 m + 2 i t \sigma^2 \hbar}} \sqrt{2\pi}}{\sqrt{\frac{1}{\sigma^2} + \frac{i t \hbar}{m}}} + \frac{R \left(e^{-\frac{2 i k m x + m x^2 \sigma^2 + i k^2 t \hbar}{2 m + 2 i t \sigma^2 \hbar}} \sqrt{2\pi}\right)}{\sqrt{\frac{1}{\sigma^2} + \frac{i t \hbar}{m}}},\ \frac{T \left(e^{\frac{2 i k m x - m x^2 \sigma^2 - i k^2 t \hbar}{2 m + 2 i t \sigma^2 \hbar}} \sqrt{2\pi}\right)}{\sqrt{\frac{1}{\sigma^2} + \frac{i t \hbar}{m}}}\right]$$

$$\text{Table}\Big[\text{Plot}\Big[\text{Abs}[\psi]^2 \ /.\ \Big\{R \to -\frac{m\,\mu}{m\,\mu - i\,k\,\hbar^2},\ T \to \frac{k\,\hbar^2}{i\,m\,\mu + k\,\hbar^2}\Big\} \ /.\ \{k \to 100,\ m \to 1,\ \mu \to 30,\ \sigma \to 1,\ \hbar \to 1\},\ \{x, -10, 10\},\ \text{PlotRange} \to \{0, 10\}\Big],\ \{t, -0.1, 0.1, 0.01\}\Big]$$





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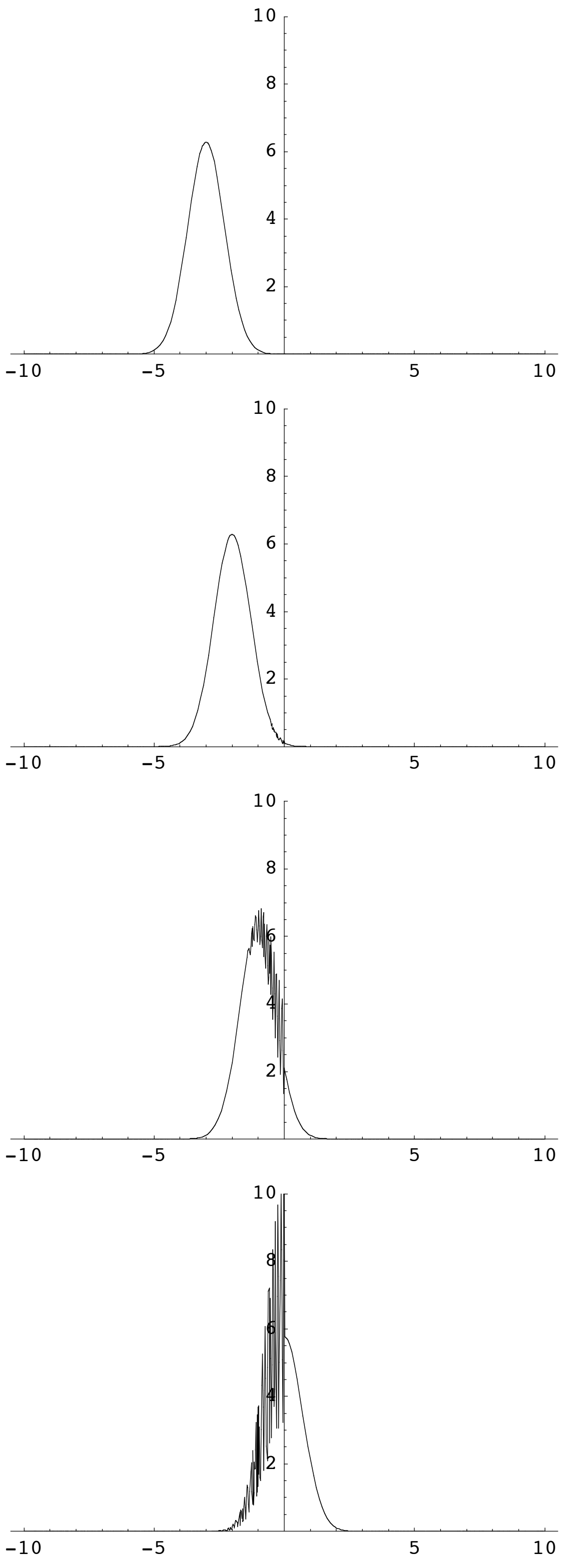
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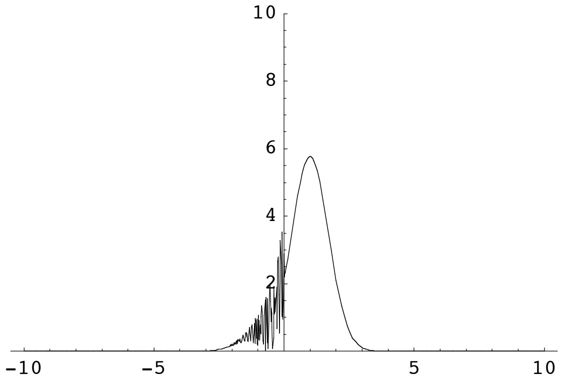
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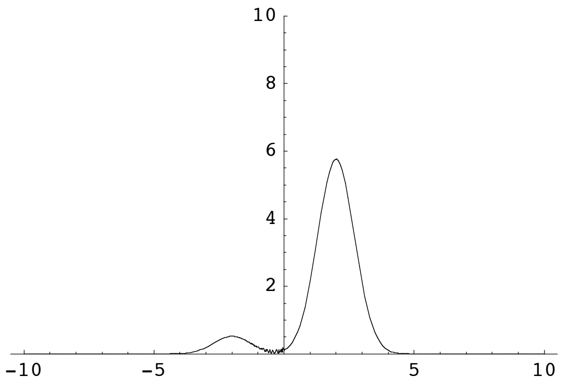
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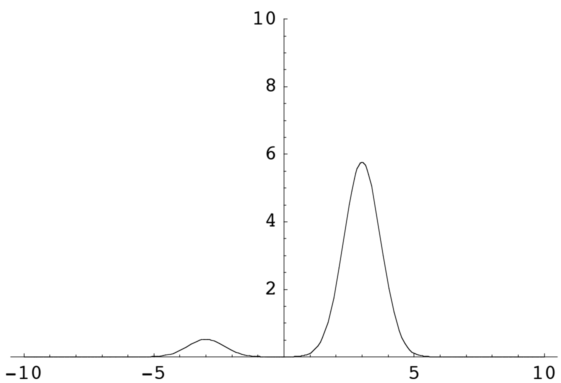
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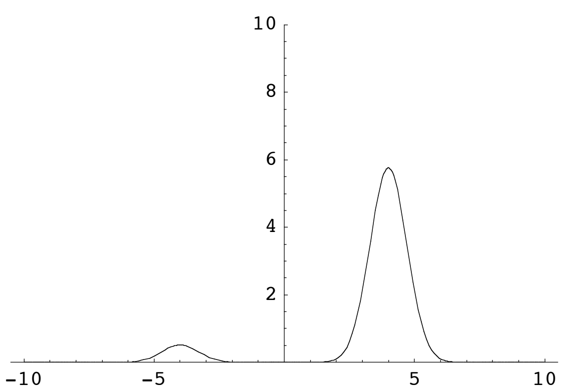
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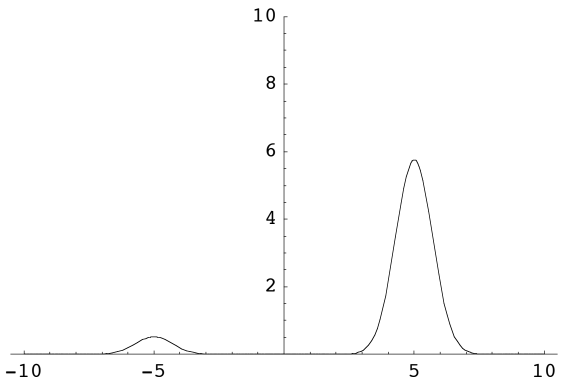
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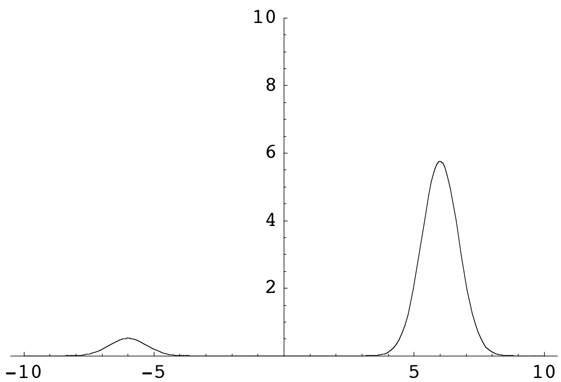
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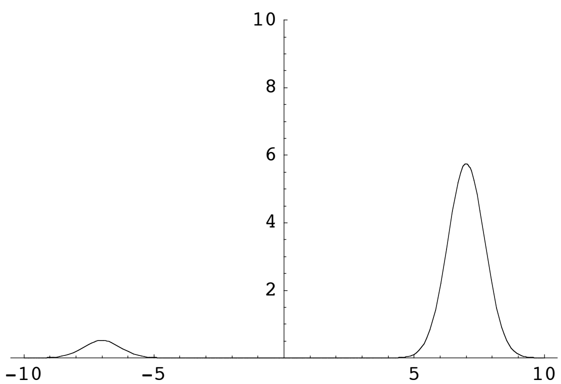
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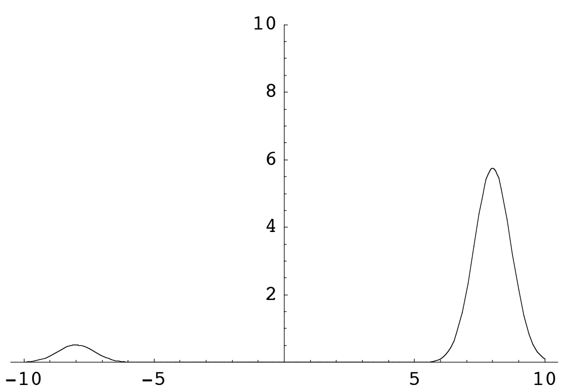
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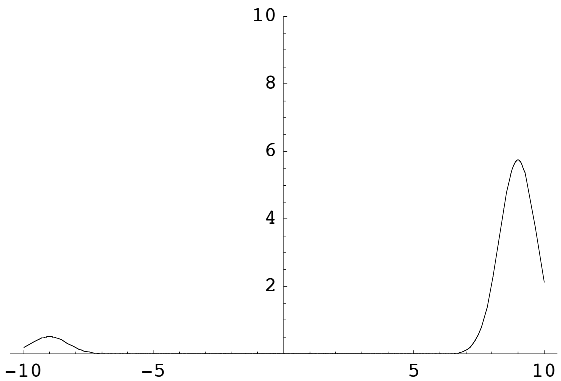

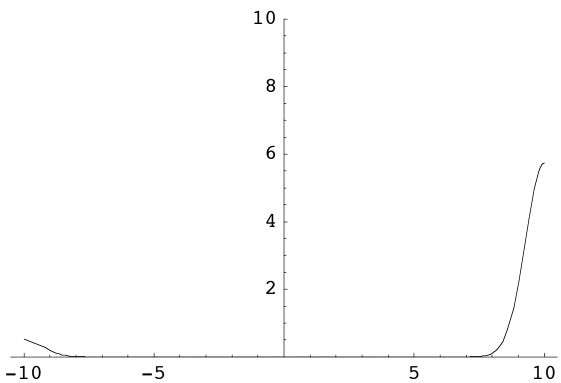


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Select the graphics and go to "Animate Selected Graphics."

(c)

Rewriting Eq. (1) into the form Eq. (3) gives $f(\pi) = -i R$, $f(0) = i(1-T)$. Therefore, the unitarity relation is rewritten as

$$1 = |R|^2 + |T|^2$$
$$= |i f(\pi)|^2 + |1 + i f(0)|^2$$
$$= |f(\pi)|^2 + 1 + i f(0) - i f^*(0) + |f(0)|^2$$
$$= \sigma + 1 - 2\,\mathrm{Im}\, f(0).$$

Therefore, $\sigma = 2\,\mathrm{Im}\, f(0)$, the one-dimensional version of the optical theorem.

2. Probability currents

The plane wave $\psi = e^{i \vec{k}\cdot\vec{x}}$ gives simply $\vec{j} = \frac{\hbar \vec{k}}{m}$, while the spherical wave $\psi = e^{ikr}/r$ gives $\vec{j} = \frac{\hbar k}{m} \frac{\vec{x}}{r^3} = -\frac{\hbar k}{m} \vec{\nabla} \frac{1}{r}$. For the former, $\vec{j}$ is constant and hence its divergence simply vanishes. For the latter, $\vec{\nabla} \cdot \vec{j} = -\frac{\hbar k}{m} \Delta \frac{1}{r} = \frac{\hbar k}{m} 4\pi\,\delta(\vec{x})$ indeed. Therefore, the spherical wave is a wave that emanates from the origin.

For fun, I plot below the probability current on the x-y plane at z=0 for both cases.

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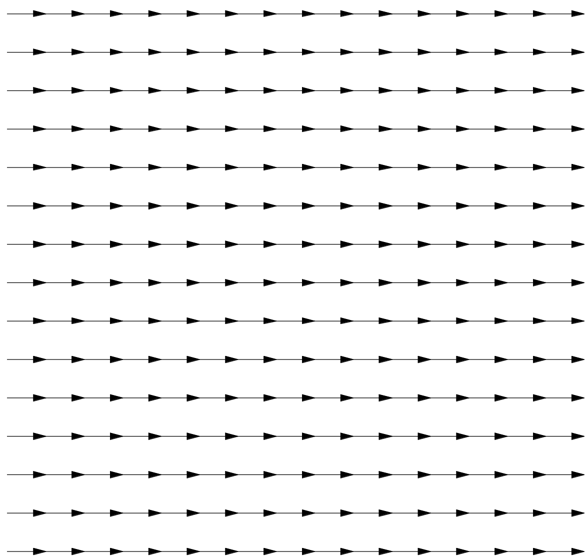
In[21]:= **PlotVectorField**[{1, 0}, {x, -1, 1}, {y, -1, 1}]



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In[22]:= **PlotVectorField**$\left[\frac{\{x, y\}}{(x^2 + y^2)^{3/2}}, \{x, -1, 1\}, \{y, -1, 1\}\right]$

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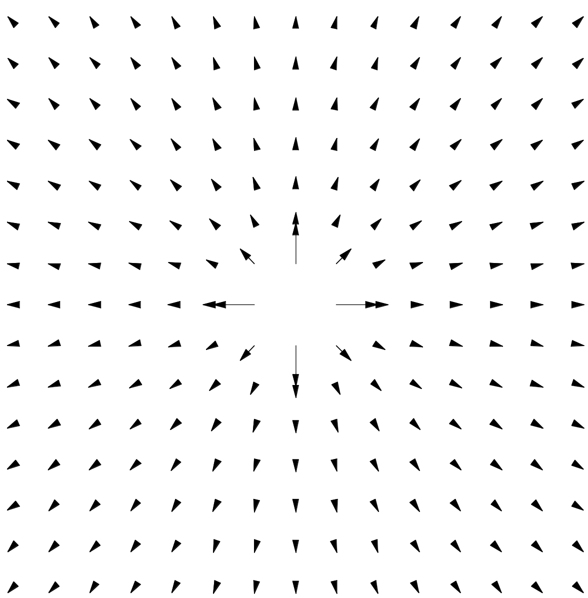

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3. Classical Hard Sphere Scattering

For the impact parameter $b > a$, the point-like particle will not get scattered by the hard sphere and merely passes it by. On the other hand, if $b < a$, it gets scattered into the angle $\theta = \pi - 2\sin^{-1}\frac{b}{a}$. Solving it, we find $b = a\cos\frac{\theta}{2}$. Because the original beam is assumed to have a uniform area density, the cross section is
$d\sigma = b\; d b\; d\phi = -a\cos\frac{\theta}{2}\, a\sin\frac{\theta}{2}\, d\theta\, d\phi = -\frac{a^2}{4}\sin\theta\, d\theta\, d\phi = \frac{a^2}{4}\, d\cos\theta\, d\phi$.
Hence the differential cross section is
$\frac{d\sigma}{d\Omega} = \frac{a^2}{4}$, and the total cross section is $\sigma = \int\frac{a^2}{4}\, d\Omega = \pi a^2$, nothing but the geometric cross section of the hard sphere.