HW #10 (221B), due Apr 15, 4pm

1. Take the Schrödinger field Lagrangian without the interaction,

$$L = \int d\vec{x} \left[\psi^* i\hbar\dot{\psi} - \frac{-\hbar^2\vec{\nabla}^2}{2m}\psi \right]. \tag{1}$$

   Rewrite the Lagrangian with the Fourier modes $\psi = \sum_{\vec{p}} a(\vec{p}) \frac{1}{L^{3/2}} e^{i\vec{p}\cdot\vec{x}/\hbar}$ with the box normalization and $\vec{p} = \hbar(n_x, n_y, n_z)/L$ for $n_x, n_y, n_z \in \mathbb{Z}$.

2. A discretized version of the Lagrangian is

$$L = \sum_i c_i^* i\hbar\dot{c}_i - \sum_{\langle i,j\rangle} t(c_j^* c_i + c_i^* c_j). \tag{2}$$

   The sum over $\langle i, j\rangle$ means it is summed over all nearest-neighbor sites of the lattice. Here, t is a parameter (not time).

   (a) Obtain the canonical commutation relation among c_i, $c_j^\dagger$ and the Hamiltonian.

   (b) Show that the one-particle states

$$\sum_k c_k^\dagger e^{ik\kappa}|0\rangle \tag{3}$$

   are eigenstates of the Hamiltonian. Assume that space is only one-dimensional for this purpose.

   (c) Show that this Lagrangian reduces to the original one in the limit of small lattice spacings.