Physics 221B: Solution to HW # 9

1) The Bogliubov Transformation

From the definition
\[ b = a \cosh \eta + a^\dagger \sinh \eta, \quad b^\dagger = a^\dagger \cosh \eta + a \sinh \eta, \]
we can easily see
\[
[b, b^\dagger] = [a, a^\dagger] \cosh^2 \eta + [a^\dagger, a] \sinh^2 \eta + ([a, a] + [a^\dagger, a^\dagger]) \sinh \eta \cosh \eta \\
= [a, a^\dagger] (\cosh^2 \eta - \sinh^2 \eta) = [a, a^\dagger] = 1.
\]

This is why the Bogliubov transformation is useful; we’ve only changed what we mean by creating and annihilating but we’ve retained the canonical commutation relation.

Our given Hamiltonian is
\[ H = \hbar \omega a^\dagger a + \frac{1}{2} V (aa + a^\dagger a^\dagger). \]

Let’s look at the simplest Hamiltonian we can construct from the \( b \) operators,
\[
\xi b^\dagger b = \xi (a^\dagger a \cosh^2 \eta + a^\dagger a^\dagger \cosh \eta \sinh \eta + aa \cosh \eta \sinh \eta + aa^\dagger \sinh^2 \eta) \\
= \xi (a^\dagger a \cosh 2\eta + (a^\dagger a^\dagger + aa) \frac{\sinh 2\eta}{2} + \sinh^2 \eta),
\]
where \( \xi \) is some constant. Notice that if
\[ \xi \sinh 2\eta = V, \quad \xi \cosh 2\eta = \hbar \omega, \]
we have reproduced our Hamiltonian up to the constant \( \sinh^2 \eta \). Playing around with hyperbolic trigonometry we find
\[ \xi = \sqrt{(\hbar \omega)^2 - V^2}, \quad \sinh^2 \eta = \frac{1}{2} \left( \frac{\hbar \omega}{\sqrt{(\hbar \omega)^2 - V^2}} - 1 \right), \]
and the Hamiltonian is
\[ H = \sqrt{(\hbar \omega)^2 - V^2} b^\dagger b - \frac{1}{2} \hbar \omega + \frac{1}{2} \sqrt{(\hbar \omega)^2 - V^2}. \]
The energy levels are
\[ E_n = \sqrt{(\hbar \omega)^2 - V^2} \left( n + \frac{1}{2} \right) - \frac{1}{2} \hbar \omega. \]
For Hilbert space operators $A$, $B$, the Hausdorff formula reads

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2!} [B, [B, A]] + \ldots.$$  

To check that $b = U a U^{-1}$ for $U = e^{(aa - a^a a^\dagger)\eta/2}$, first note

$$[(aa - a^a a^\dagger)\eta/2, a] = \eta a^\dagger \quad \text{and} \quad [(aa - a^a a^\dagger)\eta/2, a^\dagger] = \eta a.$$  

Now, use Hausdorff’s formula to write

$$U a U^{-1} = a + a^\dagger \eta + \frac{1}{2!} \eta^2 + \ldots$$

where $a$ appears with even powers of $\eta$ and $a^\dagger$ with odd powers. Factoring $a$ and $a^\dagger$ out and recognizing the power series

$$U a U^{-1} = a \cosh \eta + a^\dagger \sinh \eta = b.$$  

2) Superfluid Flow  

a)  
Look at the time independent wave $\psi = f(r)e^{in\theta}$, where $f$ is real. The number density is $\rho = \psi^* \psi = f(r)^2$. The current density, using the regular definition and the gradient in polar coordinates, is

$$\vec{j} = \frac{\hbar}{2m} \left( \psi^* \vec{\nabla} \psi - \vec{\nabla} \psi^* \psi \right) = f(r)^2 \frac{n\hbar}{nr} \hat{\theta}.$$  

The $\theta$ dependence canceled between the two terms. The velocity of the superfluid is

$$\vec{v} = \vec{j}/\rho = \frac{n\hbar}{nr} \hat{\theta}.$$  

b)  
The equation of motion we have seen in class is

$$i\hbar \dot{\psi} + \frac{\hbar^2 \nabla^2}{2m} \psi + \mu \psi - \lambda \psi^* \psi = 0.$$  

Since we are looking at a time independent solution the kinetic term vanishes. Using the polar Laplacian and dividing through by $e^{in\theta}$ gives the equation of motion for $f$

$$\frac{\hbar^2}{2m} \left( f'' + \frac{1}{r} f' - \frac{n^2}{r^2} f \right) + \mu f - \lambda f^3 = 0.$$  


Indeed, for $r \to 0$ we keep only the $O(r^{-2})$ term which shows $f(0) = 0$. If we take $r \to \infty$ we can drop the appropriate terms and assume $f$ asymptotes to a constant (meaning $f''' \to 0$) and we get $f(\infty) = \sqrt{\mu/\lambda}$.

c) The velocity is in the $\hat{\theta}$ direction and therefore circles the origin.

d) Next we calculate the circulation, $\kappa = \oint \vec{v} \cdot d\vec{l}$. Since $\vec{v}$ is in the $\hat{\theta}$ direction we have $d\vec{l} = (0, rd\theta, 0)$, and we can calculate

$$
\kappa = \oint \vec{v} \cdot d\vec{l} = \int_0^{2\pi} \frac{n\hbar}{mv} rd\theta = \frac{2\pi h}{m} n.
$$

We should point out that in order for the field $\psi = f(r)e^{in\theta}$ to be single valued, $n$ must be an integer, and therefore the circulation is quantized.