1) a)

Inserting the potential \( V(x) = \gamma \delta(x) \) into the Lippmann-Schwinger equation from HW (1) and integrating over the delta-function,

\[
\psi(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar^2 k} e^{ik|x|} \psi(0). \tag{1}
\]

Evaluating this equation at \( x = 0 \) gives an expression which we can solve for \( \psi(0) \),

\[
\psi(0) = \frac{1}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar^2 k} e^{ik|0|} \psi(0) \Rightarrow \psi(0) = \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar^2 k}{\hbar^2 k + im\gamma},
\]

so that

\[
\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}(e^{ikx} - e^{ikr} \frac{im\gamma}{\hbar^2 k + im\gamma}). \tag{2}
\]

b)

In the regions \( x > 0 \) and \( x < 0 \), the potential vanishes and \( \psi(x) \) is just a sum of same-energy plane waves which clearly satisfies the free Schrödinger equation. What’s going on at \( x = 0 \)? You might remember the condition for the discontinuity of \( \partial_x \psi \) at the location of a delta function from last semester. But if not, let's refresh our memory. We demand our wavefunction \( \psi \) will satisfy Schrödinger equation everywhere,

\[
\frac{-\hbar^2}{2m} \partial_x^2 \psi(x) + V(x) \psi(x) = E \psi(x).
\]

If \( V \) happens to be a delta function, \( V(x) = \gamma \delta(x) \), we can more easily make sense out of this equation at \( x = 0 \) by integrating both sides over a small vicinity of the origin.

\[
\int_{-\epsilon}^{+\epsilon} dx \left[ -\frac{\hbar^2}{2m} \partial_x^2 + \gamma \delta(x) \right] \psi(x) = E \int_{-\epsilon}^{+\epsilon} \psi(x) dx.
\]

In the limit \( \epsilon \to 0 \) the right hand side vanishes. We then get

\[
\left[ -\frac{\hbar^2}{2m} \partial_x \psi(x) \right]_{-0}^{+0} + \gamma \psi(0) = 0
\]

or

\[
\psi'(x)|_{-0}^{+0} = \frac{2m\gamma}{\hbar^2} \psi(0).
\]
Let's check that our solution from a) satisfies this. Reading off of equation (1)

\[
\psi'(x) = \begin{cases} 
\frac{ikx}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar k} e^{ikx} \psi(0) & \text{for } x > 0 \\
\frac{ikx}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar k} (-ik) e^{-ikx} \psi(0) & \text{for } x < 0 
\end{cases}
\]

and so

\[
\psi'(x)|^{x=0} = \frac{-im\gamma}{\hbar^2 k} ik\psi(0) - \frac{-im\gamma}{\hbar^2 k} (-ik)\psi(0) = \frac{2m\gamma}{\hbar^2} \psi(0).
\]

Alternatively, you can check that \(\psi\) satisfies the SE by careful differentiation. Notice that

\[
\partial_x e^{ik|x|} = ik\text{sign}(x)e^{ik|x|} \implies \\
\partial_x^2 e^{ik|x|} = -k^2[\text{sign}(x)]^2 e^{ik|x|} + 2iku(x)e^{ik|x|} = -k^2 e^{ik|x|} + ik\delta(x)
\]

where we've used \([\text{sign}(x)]^2 = 1\) and the fact that \(f(x)\delta(x) = f(0)\delta(x)\) (which is more precise in an integral, but nonetheless...). So just plugging eq. (2) into the SE will give

\[
-\frac{\hbar^2}{2m} \partial_x^2 \psi(x) + \gamma \delta(x) \psi(x)
= -\frac{\hbar^2}{2m} - k^2\left(e^{ikx} - e^{ikr} \frac{im\gamma}{\hbar^2 k + im\gamma}\right) + \frac{\hbar^2}{2m} \frac{1}{\sqrt{2\pi\hbar}} \frac{im\gamma}{\hbar^2 k + im\gamma} + 2iku(x)e^{ik|x|} - k^2 e^{ik|x|} + ik\delta(x)
\]

\[
= \frac{\hbar^2 k^2}{2m} \psi(x) - \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar^2 k}{\hbar^2 k + im\gamma} \delta(x) + \frac{1}{\sqrt{2\pi\hbar}} (1 - \frac{im\gamma}{\hbar^2 k + im\gamma}) \delta(x)
\]

\[
= \frac{\hbar^2 k^2}{2m} \psi(x) = E\psi(x).
\]

c) See Mathematica notebook.

d) The pole in \(f(k', k)\) is at \(k = -im\gamma/\hbar^2\), which corresponds to the real energy \(E = -m\gamma^2/2\hbar^2\). So we know there exists a stable bound state of that energy. In our derivation of the Lippmann-Schwinger equation, the only place we assume \(E > 0\), i.e. a scattering state, is when we add the incoming plane wave \(\phi(x)\) to the right hand side to satisfy our boundary conditions. We can do this because a continuum state by definition can have any energy \(> 0\); in particular we can always find a free solution \(\phi(x)\) which has the same energy as our scattering state \(\psi(x)\). This does not work for bound states which have discrete energies \(< 0\). But if we leave out \(\phi(x)\) and fix different boundary conditions, our derivation of the Lippmann-Schwinger equation holds for bound states too. That is, we can read off the bound-state wavefunction from our solution to part (a):

\[
\psi_{\text{bound}}(x) \sim e^{ikr}
\]
will be a bound-state solution when we plug in $k = -im\gamma/h^2$. Boundary conditions for a bound state are that the wavefunction decays at both infinities, which this clearly does for $\gamma < 0$. Normalizing,

$$\psi_{\text{bound}} = \sqrt{-\frac{m\gamma}{\hbar^2}} e^{m\gamma r/\hbar^2}.$$ 

It is easy to check that

$$-\frac{k^2}{2m} \frac{d^2}{dx^2} e^{m\gamma r/\hbar^2} + \gamma \delta(x) e^{m\gamma r/\hbar^2} = -\frac{m\gamma^2}{2\hbar^2} e^{m\gamma r/\hbar^2}.$$

e)

Plugging $V(x) = \gamma \delta(x)$ into the 3-d Lippmann-Schwinger equation gives

$$\psi(x) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikx} - \frac{2m}{\hbar^2} \frac{e^{ik|x|}}{4\pi|x|} \gamma \psi(0). \tag{3}$$

To avoid singularity at the origin we require $\psi(0) = 0$, but then the scattering term vanishes, and we are left with the free plane wave

$$\psi(x) = \begin{cases} \frac{1}{(2\pi\hbar)^{3/2}} e^{ikx}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

the singularity in the potential allowing the discontinuous wavefunction. This is not exactly the planewave we would have gotten if the potential was zero everywhere. In that case we get $\psi(x) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikx}$. However, when doing a scattering experiment we certainly won’t notice the change, so we can say that a delta function potential does not scatter in 3 dimensions. Note that eq. (3) can still be consistent at the origin only if $\psi$ goes to zero like $\frac{1}{\sqrt{2\pi\hbar^2}} |x|$. Assuming this is the case, writing eq. (3) for $x \to 0$ indeed gives

$$\psi(0) = \frac{1}{(2\pi\hbar)^{3/2}} - \frac{1}{(2\pi\hbar)^{3/2}} = 0.$$
2) Yukawa potential in the Born approximation

\textbf{a)}

By straightforward integration in the Born approximation (see e.g. lecture notes 2),

\[ f^{(1)}(\vec{k}, \vec{k}') = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int d\vec{x} V(\vec{x}) e^{i\vec{k}' \cdot \vec{x}} \]

\[ = -\frac{2m}{\hbar^2} \frac{1}{q} \int_0^\infty dr r^2 V_0 e^{-r/a} \sin qr \]

\[ = -\frac{2mV_0}{\hbar^2} \frac{1}{q} \left( e^{iqr-a/2} - e^{-iqr+a/2} \right) \]

\[ = -\frac{2mV_0}{\hbar^2} \frac{1}{q^2+1/a^2}, \]

where \( q^2 = |\vec{k} - \vec{k}'|^2 = 2k^2(1 - \cos \theta) \). Therefore,

\[ \sigma = \int \frac{d\sigma}{d\Omega} = \int 2\pi \left( \frac{mV_0}{\hbar^2 k^2} \right)^2 \frac{d\cos \theta}{(1 - \cos \theta + 1/2k^2a^2)^2} \]

\[ = 2\pi \left( \frac{mV_0}{\hbar^2 k^2} \right)^2 \frac{1}{1 - \cos \theta + 1/2k^2a^2} \]

\[ = 4\pi a^2 \frac{m^2 V_0^2 a^2}{\hbar^4} \frac{1}{1 + 4k^2a^2}. \]

\textbf{b)}

As discussed in lecture, we require that the difference between the true wavefunction \( \psi \) and the free plane wave \( \phi \) be small where the potential is large. We compute in the Born approximation at the origin (relabeling \( \vec{x}' \rightarrow \vec{x} \)):

\[ |\psi - \phi| \sim \frac{2m}{\hbar^2} \left| \int d\vec{x} \frac{e^{ikr}}{4\pi r} V_0 e^{-r/a} e^{ikz} \right| \ll 1. \]

The integral is much easier to do if you integrate the \( r \) variable first (but some people managed it the other way around with the help of Mathematica or a table of integrals).

\[ \frac{2m}{\hbar^2} \int 2\pi d\cos \theta r^2 dr \frac{e^{ikr}}{4\pi r} V_0 e^{-r/a} e^{ikr \cos \theta} = \frac{mV_0}{\hbar^2} \int d\cos \theta \frac{-1}{ik + ik \cos \theta - 1/a} \]

\[ = -\frac{mV_0}{ik\hbar^2} \log(\cos \theta + 1 - 1/ika) \bigg|_1 = -\frac{mV_0}{ik\hbar^2} \log(1 - 2ika). \]

The condition for the validity of the Born approximation is

\[ \frac{mV_0}{k\hbar^2} |\log(1 - 2ika)| \ll 1. \]
Recalling the formula for the log of an imaginary number and squaring for later convenience (even though squaring weakens the meaning of $\ll$) we get

$$\frac{m^2 V_0^2}{k^2 h^2} \ll \frac{1}{[\log(1 + 4k^2a^2)]^2 + [\arctan(2ka)]^2}.$$  (4)

We can see that its easier satisfying the above for large $k$, where arctan ceases to increase and $k$ increases faster than its log. Alternatively, for a given momentum we can satisfy (4) by decreasing the strength of the potential $V_0$ or the potential range $a$.

(c)

The combination appearing on the left hand side of (4) happens to appear in the total cross-section as well. This can be used to get an upper bound on $\sigma$.

$$\sigma = 4\pi a^2 \frac{m^2 V_0^2 a^2}{h^4} \frac{4}{1 + 4k^2a^2} \ll 4\pi a^2 \frac{4}{1 + 4k^2a^2} \frac{k^2a^2}{[\log(1 + 4k^2a^2)]^2 + [\arctan(2ka)]^2}$$

If the huge mess that is multiplying $4\pi a^2$, which we will call $f(ka)$, happens to be smaller or equal to 1 then we’ve proved $\sigma \ll 4\pi a^2$. ($4\pi a^2$ is used as the geometric cross-section, and not just $\pi a^2$. As we saw in lecture this can be heuristically justified by thinking about the smearing of the incoming wave-packet compare to the classical particle.)

The Mathematica notebook below plots $f(ka)$ for values of $0 < ka < 10$ to show that indeed $f(ka) \leq 1$. Plotting for larger intervals shows the same. Some of you preferred an analytic proof and showed that $f(x) \leq 1$ by showing that $f(0) = 1$ and $f'(x > 0)$ is always negative.

```
in[1]:= f[x_] := 4 x^2 / (1 + 4 x^2) 1 / [log[1 + 4 x^2]^2 + arctan[2 x]^2]
in[3]:= Plot[f[x], {x, 0, 10}]
```