Physics 221B: Solution to HW #10

1) The Electromagnetic Field and its Hamiltonian

a)
This is a standard computation which can be found in most books on quantum field theory, though perhaps in the context of the scalar Klein-Gordon field.

\[ H = \frac{1}{8\pi} \int d\vec{x} \vec{E}^2 + \vec{B}^2. \]

Using \( \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \) and plugging in the mode expansion for \( \vec{A} \), the \( \vec{E}^2 \) contribution to the energy is

\[ \int d\vec{x} \vec{E}^2 = \int d\vec{x} \frac{2\pi\hbar c^2}{L^3} \sum_{\vec{p},\vec{q},\lambda,\lambda'} (-i)^2 \sqrt{\omega_{\vec{p}}\omega_{\vec{q}}} (\epsilon^\dagger_\lambda(\vec{p})a_\lambda(\vec{p})e^{i\vec{q} \cdot \vec{x}/\hbar} - \epsilon^\dagger_\lambda(\vec{p})^*a_{\lambda'}(\vec{p})^*e^{-i\vec{q} \cdot \vec{x}/\hbar}) \\ \quad \times (\epsilon^\dagger_{\lambda'}(\vec{q})a_{\lambda'}(\vec{q})e^{i\vec{q} \cdot \vec{x}/\hbar} - \epsilon^\dagger_{\lambda'}(\vec{q})^*a^\dagger_{\lambda'}(\vec{q})^*e^{-i\vec{q} \cdot \vec{x}/\hbar}). \]

After multiplying out, rewrite

\[ \int d\vec{x} e^{i(\vec{p} \cdot \vec{x} - \vec{q} \cdot \vec{x})/\hbar} \to (2\pi\hbar)^3 \delta^3(\vec{p} \pm \vec{q}) \]

\[ \sum_{\vec{q}} \to \frac{L^3}{(2\pi\hbar)^3} \int d\vec{q}. \]

Then since \( \omega_{-\vec{p}} = \omega_{\vec{p}} \), after carrying out the obvious integrals we have

\[ \int d\vec{x} \vec{E}^2 = -\sum_{\vec{p}} 2\pi\hbar \omega_{\vec{p}} \sum_{\lambda,\lambda'} (\epsilon^\dagger_\lambda(\vec{p})a_\lambda(\vec{p})\epsilon^\dagger_{\lambda'}(-\vec{p})a_{\lambda'}(-\vec{p}) - \epsilon^\dagger_\lambda(\vec{p})^*a_{\lambda'}^\dagger(\vec{p})\epsilon^\dagger_{\lambda'}(-\vec{p})a_{\lambda'}(\vec{p}) \]

\[ - \epsilon^\dagger_\lambda(\vec{p})a_\lambda(\vec{p})\epsilon^\dagger_{\lambda'}(-\vec{p})a_{\lambda'}(-\vec{p}) + \epsilon^\dagger_{\lambda'}(\vec{p})a_{\lambda'}(\vec{p})\epsilon^\dagger_{\lambda'}(\vec{p})^*a_{\lambda'}^\dagger(-\vec{p}). \]

Now

\[ \epsilon^\dagger_\lambda(\vec{p})\epsilon^\dagger_{\lambda'}(-\vec{p}) = -\delta_{\lambda,\lambda'} \]

\[ \epsilon^\dagger_{\lambda'}(-\vec{p})\epsilon^\dagger_{\lambda}(\vec{p}) = \delta_{\lambda,\lambda'}, \]

with analogous results for the other combinations (check simple cases). Then

\[ \int d\vec{x} \vec{E}^2 = \sum_{\vec{p},\lambda} 2\pi\hbar \omega_{\vec{p}} (a_\lambda(\vec{p})a_{\lambda}(-\vec{p}) + a_{\lambda'}(\vec{p})a_{\lambda'}(-\vec{p})) + a_{\lambda'}^\dagger(\vec{p})a^\dagger_{\lambda'}(-\vec{p}). \]

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1I thank Ed Boyda once more.
The terms like $aa$ and $a\dagger a\dagger$ cancel with similar terms from $\vec{B}^2$ while the other terms add. Including the $1/8\pi$ from the definition of energy,

$$H = \frac{1}{8\pi} \int d\vec{x} \vec{E}^2 + \vec{B}^2 = \frac{1}{2} \sum_{\vec{p},\lambda} \hbar \omega_\vec{p} (a_\lambda^\dagger(\vec{p})a_\lambda(\vec{p}) + a_\lambda(\vec{p})a_\lambda^\dagger(\vec{p})).$$

Using $[a, a\dagger] = 1$ gives the result

$$H = \sum_{\vec{p},\lambda} \hbar \omega_\vec{p} (a_\lambda^\dagger(\vec{p})a_\lambda(\vec{p}) + \frac{1}{2}).$$

b)

We consider the coherent state of photons with $\vec{p} = (0, 0, p)$ and helicity $\lambda = +$.

$$|f, t\rangle := e^{-\frac{f^* f}{2}} e^{e^{-ic|\vec{p}|^t/h} a_\lambda^\dagger(\vec{p}) |0\rangle},$$

$$i\hbar \frac{\partial}{\partial t} |f, t\rangle = c |\vec{p}| f e^{-ic|\vec{p}|^t/h} a_\lambda^\dagger(\vec{p}) |f, t\rangle.$$

Since $|f, t\rangle$ is an eigenstate of the annihilation operator, $a_\lambda(\vec{q}) |f, t\rangle = \delta_{\lambda+} \delta_{\vec{p}\vec{q}} f e^{-ic|\vec{p}|^t/h} |f, t\rangle$,

$$H |f, t\rangle = \sum_{\vec{q},\lambda} c |\vec{q}| a_\lambda^\dagger(\vec{q})a_\lambda(\vec{q}) |f, t\rangle = c |\vec{p}| a_\lambda^\dagger(\vec{p}) f e^{-ic|\vec{p}|^t/h} |f, t\rangle,$$

ignoring the zero point energy and using the delta functions to perform the sums. Clearly $i\hbar \frac{\partial}{\partial t} |f, t\rangle = H |f, t\rangle$.

c)

Again, $|f, t\rangle$ is an eigenstate of the annihilation operator and $\langle f, t |$ is an eigenstate of the creation operator so that

$$\langle f, t | a_\lambda(\vec{q}) |f, t\rangle = \delta_{\lambda+} \delta_{\vec{p}\vec{q}} f e^{-ic|\vec{p}|^t/h},$$

$$\langle f, t | a_\lambda^\dagger(\vec{q}) |f, t\rangle = \delta_{\lambda+} \delta_{\vec{p}\vec{q}} f^* e^{ic|\vec{p}|^t/h}.$$

The definition of $\vec{A}$ gives immediately

$$\langle f, t | \vec{A} |f, t\rangle = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \frac{1}{\sqrt{\omega_\vec{p}}}(\vec{\varepsilon}_+(\vec{p}) f e^{-ip\cdot x/h} + \vec{\varepsilon}_+(\vec{p}) f^* e^{ip\cdot x/h}),$$

where $p \cdot x = c |\vec{p}| t - \vec{p} \cdot \vec{x}$ is the Minkowski scalar product. The coherent state expectation value reproduces a classical plane wave.
3)  

a)  
Work in units where \( \hbar = 1 \). It is convenient to rewrite the Hamiltonian as  

\[
H = -J \sum_{\langle ij \rangle} \vec{s}_1 \cdot \vec{s}_2 = -J \sum_{\langle ij \rangle} s_{zi}s_{zj} + \frac{1}{2} (s_{+i}s_{-j} + s_{-i}s_{+j})
\]

where \( s_{\pm} = s_x \pm is_y \). When all spins are up along the \( z \) axis only the first term in \( H \) contributes because the other two terms will “try to raise” spins that are already up. Therefore, defining \( |0\rangle \equiv |\uparrow\uparrow\uparrow\uparrow \ldots \rangle \)

\[
H |0\rangle = -J \sum_{\langle i,j \rangle} s_{zi}s_{zj} |0\rangle = -J \sum_{\langle i,j \rangle} \frac{1}{4} |0\rangle = -\frac{NJ}{4} |0\rangle
\]

where \( N \) the number of pairs.

b)  
The system is rotationally invariant, so the Hamiltonian should commute with the rotation operator. We can check this for the particular rotation \( \tilde{U} = \Pi_i U(\theta) = e^{-i\theta \sum_i s_{yi}} : \)

\[
[s_{yi} + s_{yj}, \vec{s}_i \cdot \vec{s}_j] = [s_{yi} + s_{yj}, s_{xi}s_{xj} + s_{yi}s_{yj} + s_{zi}s_{zj}]
\]

\[
= -is_{zi}s_{xj} - is_{xi}s_{zj} + is_{xi}s_{zj} + is_{zi}s_{xj} = 0.
\]

Commuting operators have commuting exponentials, so \( H\tilde{U} = \tilde{U}H \); the Hamiltonian is invariant under the rotation. This means that the new ground state \( |0\rangle := \tilde{U}|0\rangle \) satisfies

\[
H\tilde{U}|0\rangle = \tilde{U}H|0\rangle = E_0\tilde{U}|0\rangle,
\]

so the rotated state is also a ground state, an equivalent “choice” for the spontaneous symmetry breaking.

We want to check that the two ground states are orthogonal in the limit \( N \to \infty \) where \( N \) is the number of spins. Consider a given spin which in the ground state is in the state \( |\uparrow\rangle = |0\rangle \). The rotation sends this to

\[
|\uparrow\rangle = \left( \begin{array}{c} \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \cos \frac{\theta}{2} |\uparrow\rangle + \sin \frac{\theta}{2} |\downarrow\rangle.
\]
Taking the inner product $(0|0')$ will give a product of factors $(↑ | ↑')$, one for each spin. The factors are
\[
\langle ↑ | ↑' \rangle = \langle ↑ | (\cos \frac{\theta}{2} ↑) + \sin \frac{\theta}{2} ↓ \rangle = \cos \frac{\theta}{2}.
\]
For $N$ spins,
\[
\langle 0|0' \rangle = (\cos \frac{\theta}{2})^N.
\]
For any non-zero rotation, the factor $\cos \frac{\theta}{2}$ will be less than one. Thus as $N \to \infty$, $(\cos \frac{\theta}{2})^N \to 0$.

c)
Now we consider the state
\[
|\psi\rangle = \sum_n e^{ika} |↑↑↑↓n↑↑...⟩.
\]
This time when we act $H$ on $|\psi\rangle$ the last two terms in $H$ may contribute. Defining $|\psi_n\rangle \equiv |↑↑↓n↑...⟩$ we see how $H$ acts
\[
H|\psi_n\rangle = -J \frac{N-4}{4} |\psi_n\rangle - \frac{J}{2} (|\psi_{n-1}\rangle + |\psi_{n+1}\rangle).
\]
The first term above is just the ground state energy form part (a), but after two pairs have changed from $s_z i s_z j = +1$ to $s_z i s_z j = -1$. The rest comes from the $s_+ s_-$ terms "moving" the spin that points down by one site to the left or to the right.

Now, for $|\psi\rangle = \sum_n e^{ika} |\psi_n\rangle$ we get
\[
H|\psi\rangle = -J \sum_n e^{ika} \frac{(N-4)}{4} |\psi_n\rangle - \frac{J}{2} \sum_n e^{i(n+1)ka} |\psi_n\rangle - \frac{J}{2} \sum_n e^{i(n-1)ka} |\psi_n\rangle
\]
\[
= -J \left( \frac{N-4}{4} - \frac{1}{2} e^{ika} - \frac{1}{2} e^{-ika} \right) \sum_n e^{ina} |\psi_n\rangle = -J \left( \frac{N}{4} + 1 - \cos ka \right) |\psi\rangle.
\]
The excitation energy is obviously
\[
\Delta E = J(1 - \cos ka).
\]
This is a tiny excitation for $N \gg 1$, as we expect.