HW #9 (221B), due Apr 19, 5pm

1. Suppose annihilation and creation operators satisfy the standard commutation relation \([a, a^\dagger] = 1\). Show that the Bogliubov transformation

\[
b = a \cosh \eta + a^\dagger \sinh \eta
\]

preserves the commutation relation of creation and annihilation operators \([b, b^\dagger] = 1\). Use this fact to obtain eigenvalues of the following Hamiltonian

\[
H = \hbar \omega a^\dagger a + \frac{1}{2} V(aa + a^\dagger a^\dagger).
\]

(There is an upper limit on \(V\) for which this can be done). Also show that the unitarity operator

\[
U = e^{(aa - a^\dagger a^\dagger)\eta/2}
\]

can relate two set of operators \(b = UaU^{-1}\).

2. We can discuss macroscopic motions of the superfluid by regarding \(\psi(\vec{x}, t)\) as a classical wave. We are particularly interested in time-independent and \(z\)-independent solutions of the form

\[
\psi(x, y) = f(r)e^{i\theta},
\]

where \(f(r)\) is a real function. Answer the following questions.

(a) Write down the velocity field \(\vec{v} = \vec{j}/\rho\) using the number density \(\rho = \psi^* \psi\) and the momentum density \(\vec{j} = \frac{\hbar}{2m}(\psi^* \vec{\nabla} \psi - \vec{\nabla} \psi^* \psi)\).

(b) Write down the equation of motion

\[
i\hbar \dot{\psi} + \frac{\hbar^2 \Delta}{2m} \psi + \mu \psi - \lambda \psi^* \psi \psi = 0
\]

in terms of \(f(r)\).

Note This equation allows a monotonic solution with \(f(0) = 0\) and \(f(\infty) = \sqrt{\mu/\lambda}\). This solution is called a vortex solution.

(c) Show that the velocity field circles around the origin.

(d) Show that the circulation defined by \(\kappa = \oint \vec{v} \cdot d\vec{l}\) is quantized.