HW #9 (221B), due Apr 19, 5pm

1. Suppose annihilation and creation operators satisfy the standard commutation relation $[a, a^\dagger] = 1$. Show that the Bogliubov transformation

$$b = a \cosh \eta + a^\dagger \sinh \eta \tag{1}$$

preserves the commutation relation of creation and annihilation operators $[b, b^\dagger] = 1$. Use this fact to obtain eigenvalues of the following Hamiltonian

$$H = \hbar\omega a^\dagger a + \frac{1}{2} V (aa + a^\dagger a^\dagger). \tag{2}$$

(There is an upper limit on V for which this can be done). Also show that the unitarity operator

$$U = e^{(aa - a^\dagger a^\dagger)\eta/2} \tag{3}$$

can relate two set of operators $b = UaU^{-1}$.

2. We can discuss macroscopic motions of the superfluid by regarding $\psi(\vec{x}, t)$ as a classical wave. We are particularly interested in time-independent and z-independent solutions of the form

$$\psi(x, y) = f(r) e^{in\theta}, \tag{4}$$

where $f(r)$ is a real function. Answer the following questions.

   (a) Write down the velocity field $\vec{v} = \vec{\jmath}/\rho$ using the number density $\rho = \psi^* \psi$ and the momentum density $\vec{\jmath} = \frac{\hbar}{2mi}(\psi^* \vec{\nabla} \psi - \vec{\nabla} \psi^* \psi)$.

   (b) Write down the equation of motion

$$i\hbar\dot{\psi} + \frac{\hbar^2 \Delta}{2m} \psi + \mu\psi - \lambda \psi^* \psi \psi = 0 \tag{5}$$

   in terms of $f(r)$.

   Note This equation allows a monotonic solution with $f(0) = 0$ and $f(\infty) = \sqrt{\mu/\lambda}$. This solution is called a vortex solution.

   (c) Show that the velocity field circles around the origin.

   (d) Show that the cirulation defined by $\kappa = \oint \vec{v} \cdot d\vec{l}$ is quantized.

   Note A n-vortex actually breaks up into n single vortices to lower the energy. Look at pictures of vortices in regular arrays in rotating superfluid Helium in a paper by E.J. Yarmchuk, M.J.V. Gordon, and R.E. Packard, "Observation of Stationary Vortex Arrays in Rotating Superfluid Helium," *Phys. Rev. Lett.* **43**, 214-217 (1979).