# HW #9 (221B), due Apr 20, 4pm

1. Using the mode expansion of the Maxwell field, show that the Hamiltonian can be written as

$$H = \int d\vec{x} \frac{1}{8\pi} \left( \vec{E}^2 + \vec{B}^2 \right) = \sum_{\vec{p}} \sum_{\lambda} c|\vec{p}| \left( a_\lambda^\dagger(\vec{p}) a_\lambda(\vec{p}) + \frac{1}{2} \right). \tag{1}$$

2. Consider a coherent state of photons in a particular momentum $\vec{p} = (0, 0, p)$ and helicity $+1$

$$|f\rangle = e^{-f^* f/2} e^{f a_+^\dagger(\vec{p})} |0\rangle. \tag{2}$$

Show that (1) the Schrödinger equation $i\hbar \frac{\partial}{\partial t}|f\rangle = H|f\rangle$ has a solution $|f, t\rangle = |fc^{-ic|\vec{p}|t/\hbar}\rangle$, and (2) calculate the expectation value of the Maxwell field $\langle f, t|\vec{A}(\vec{x})|f, t\rangle$. You can see that this state describes a classical electromagnetic wave such as laser.

3. Consider spins 1/2 on a lattice with nearest neighbor interaction for a ferromagnet

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j. \tag{3}$$

Here, $i, j$ refer to lattice sites and $\langle i, j \rangle$ are nearest neighbor pairs. Answer the following questions. You can use the identity

$$U(\theta) = e^{-i\theta \vec{s}_y} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{4}$$

   (a) Show that the state with all spins up is an eigenstate of the Hamiltonian (it is actually the ground state).

   (b) Show that $U(\theta)$ acting on all spins at the same time give you another ground state which is orthogonal to the previous one in the limit of infinite number of spins.

   (c) Consider a site-dependent rotation of the ground state

$$\prod_i (U(\theta_i)| \uparrow_i\rangle). \tag{5}$$

   Show that the expectation value of the Hamiltonian for this state is the ground-state energy plus a term proportional to $\sum_{\langle i,j \rangle} (\theta_i - \theta_j)^2$ neglecting terms higher order in $\theta$. (This is the spin wave. Quantized version of it gives you "magnon" excitations of ferromagnets.)