HW #5 Solutions (221B)

1) α particles are sensitive and small and fast

This problem follows straightforward procedure—integrate, plug in—but some routes are straighter than others. If you come across a charge measured in Coulombs in a nuclear physics problem, you are on a decidedly crooked path.

We take a phenomenological model where the α particle lives in a nucelear potential described (in cgs units) by

$$V = \begin{cases} -V_0, & r < a \\ \frac{2Z'e^2}{r}, & r > a. \end{cases}$$

The interior potential well models a strong nuclear binding force, the exterior potential is the Coulomb repulsion between the daughter nucleus ($Z' = Z-2$ after α decay) and the escaping α particle.

In the WKB approximation the transmission coefficient is

$$T = \exp -2 \int_a^b dr \sqrt{\frac{2m(V(r) - E)}{\hbar^2}},$$

where a and $b = \frac{2Z'e^2}{E}$ are the classical turning points. We can consider only the s-wave contribution because higher-order partial waves will be suppressed even further: Mathematically, the argument of the exponential scales as the area under the potential curve, and contributions $l(l+1)/r^2$ will increase this area. Physically, a particle wasting some of its energy in orbital motion has less outward-going energy to help penetrate the potential barrier. Thus

$$T = \exp\Big\{ -2\sqrt{\frac{2m}{\hbar^2}} \int_a^b dr \sqrt{\frac{2Z'e^2}{r} - E}\Big\} = \exp\Big\{ -2\sqrt{\frac{2m}{\hbar^2}}\sqrt{2Z'e^2} \int_a^b dr \sqrt{\frac{1}{r} - \frac{1}{b}}\Big\}.$$

The integral gives

$$b^{\frac{1}{2}}\Big\{ \arccos\Big(\frac{a}{b}\Big)^{\frac{1}{2}} - \Big(\frac{a}{b} - \frac{a^2}{b^2}\Big)^{\frac{1}{2}}\Big\}.$$

With the specified parameters, b is of order a, so there is no useful expansion.

$$T \approx \exp\Big\{ -\frac{4Z'e^2}{\hbar}\sqrt{\frac{2m}{E}}\Big(\arccos\Big(\frac{aE}{2Z'e^2}\Big)^{\frac{1}{2}} - \Big(\frac{aE}{2Z'e^2} - \frac{a^2E^2}{4Z'^2e^4}\Big)^{\frac{1}{2}}\Big)\Big\}.$$

Now we plug in. Rewrite the nasty expression in parentheses as

$$\Big(\arccos\Big(\frac{E}{E_0}\Big)^{\frac{1}{2}} - \Big(\frac{E}{E_0} - \frac{E^2}{E_0^2}\Big)^{\frac{1}{2}}\Big),$$

where I have identified a characteristic electric energy $E_0 := \frac{2Z'e^2}{a} \approx 51.8$ MeV, which you can compute easily by scaling up the characteristic atomic energy $e^2/a_B \approx 27.2$ eV by a factor a_B/a ($a_B \approx .529$ angstroms is the Bohr radius). The factor out front can be reorganized as

$$\frac{4Z'e^2}{\hbar}\sqrt{\frac{2m}{E}} = \frac{4Z'e^2}{\hbar c}\sqrt{\frac{2mc^2}{E}} = -4Z'\alpha\sqrt{\frac{2mc^2}{E}},$$

where $\alpha := \frac{e^2}{\hbar c} \approx 1/137$ is the fine structure constant (no relation to the α particle). Thus

$$T \approx \exp\Big\{ - 4Z'\alpha\sqrt{\frac{2mc^2}{E}}\Big(\arccos\Big(\frac{E}{E_0}\Big)^{\frac{1}{2}} - \Big(\frac{E}{E_0} - \frac{E^2}{E_0^2}\Big)^{\frac{1}{2}}\Big)\Big\}.$$

Now everything is easy to deal with. We know $Z' = Z - 2 = 90$, $\alpha \approx 1/137$, and $mc^2 = 4 \times m_p c^2 \approx 4 \times 938$ MeV, and we computed $E_0 \approx 51.8$ MeV. Therefore

$$T \approx \begin{cases} e^{-295} \approx 10^{-128} & E = 1 \text{ MeV} \\ e^{-144} \approx 10^{-62.5} & E = 3 \text{ MeV} \\ e^{-51.9} \approx 10^{-22.5} & E = 10 \text{ MeV} \\ e^{-8.82} \approx 10^{-3.83} & E = 30 \text{ MeV}. \end{cases}$$

The salient feature of these numbers is their extreme sensitivity to energy. We can use them to estimate lifetimes with this cheap method: The probability for emission of an α particle goes as the transmission coefficient, so the decay rate will go as the transmission coefficient times inverse of the characteristic time scale of the nuclear decay process. I.e. the lifetime τ, which is the inverse of the decay rate, will go as the nuclear time scale divided by T. Very roughly, the time scale is $\sim \sqrt{ma^2/E} \approx 10^{-21}/\sqrt{E}$ seconds (with E given in MeV), so that

$$\tau \sim 10^{-21}\frac{1}{\sqrt{E\,(\text{MeV})}}\frac{1}{T} \text{ seconds}.$$

This formula gives

$$\tau \approx \begin{cases} 10^{107} & E = 1 \text{ MeV} \\ 10^{41} & E = 3 \text{ MeV} \\ 10^{1} & E = 10 \text{ MeV} \\ 10^{-18} & E = 30 \text{ MeV}. \end{cases}$$

These numbers are striking again because of their 125-orders-of-magnitude range over 1 order of magnitude in energy. Mostly what this tells us is that if we want to compute the actual lifetime of Uranium, we should spend most of our effort determining the range of energies of a typical decay. My quantum mechanics textbook tells me that the halflife of Uranium is of order 10^{17} seconds, so the typical energy of decays must be between 3 and 10 MeV, but we can't say much more than that. For $E \lesssim 3$ MeV Uranium would never decay, the lifetimes being many orders of magnitude longer than the age of the universe. For $E \sim 30$ MeV, Uranium would decay 10 or so orders of magnitude faster than the fastest α emitter.

Less than half the class found the correct numerical transmission coeffecients, whereas almost everyone derived the correct analytical expression. One problem was the use of $Z = 92$ instead of $Z' = Z - 2$. The other reason, I think, is that almost everyone plugged in Joules and Coulombs or ergs and esu's or whatever their favorite macroscopic system of units is. These units were invented by us big, lumbering folk who move at about a meter or a centimeter per second and weigh 50 or 100 kilograms and measure the static electricity on a pith ball. These units are just not appropriate for use with electrons or protons or other small and fast creatures. What is "appropriate"? Appropriate units are those in which the characteristic quantities in your problem are order 1. If you have to plug in $e = 1.602 \times 10^{-19}$ C, that's a big sign you are being to humancentric. You're welcome to experiment with plugging in those unnatural numbers, but taking this HW as a preliminary data set, the experiment is a failure. The other problem with using unnatural units is that it can obscure the meaning of an expression. For instance, α appears often in quantum mechanics as the expansion parameter in a perturbation series. Keep track of the α's and you will know the relative significance of different terms.

Elementary particles are often relativistic or nearly relativistic and have actions of order $\hbar$, so we choose units in which $c = 1$ and $\hbar = 1$. In these units $e^2 = \alpha$ and masses (as energies) are automatically measured in eV. We could have immediately identified the key factor in T as[1]

$$4Z'e^2\sqrt{\frac{2m}{E}} \to -4Z'\alpha\sqrt{\frac{2m}{E}},$$

without going through the intermediate step of reorganizing c's and $\hbar$'s. Here $m = 4 \times m_p = 4 \times 938$ MeV. At the end of the day, you can restore

[1]Strictly speaking the α particles in this problem are not relativistic (usually taken to mean $E \gtrsim mc^2$), but they are much closer to relativistic than to mks or cgs speeds. $\hbar = c = 1$ is still a convenient choice of units here.

your favorite units by remembering that the fundamental unit of speed is c, so a speed of, say, .954, becomes $.954c$ in cgs or mks units.

For atomic and molecular problems, a fundamental length scale is a_B, a fundamental speed is αc, and a fundamental mass is the electron mass m_e. So set $a_B = \alpha c = m_e = 1$. (How many things can you set to 1 and still be consistent? Three, if you are dealing with problems in which the units are length, time, and mass. If you added, say, temperature, you could also set the Bolzmann constant to 1). Note that since $a_B = \frac{\hbar^2}{e^2 m_e}$ and $\alpha c = \frac{e^2}{\hbar}$, our choice of units also forces $e = \hbar = 1$. This is a fortuitous coincidence, and should be contrasted with the relativistic units where $e^2 \approx 1/137$. Thus we have the following correspondance between atomic units on the left and your favorite units on the right (with a few examples in parentheses):

- 1 unit of length $\leftrightarrow a_B$ ($\approx .529$ angstroms)
- 1 unit of mass $\leftrightarrow m_e$
- 1 unit of speed $\leftrightarrow \alpha c$ ($\approx 2.12 \times 10^8$ cm/s)
- 1 unit of charge $\leftrightarrow e$
- 1 unit of energy $\leftrightarrow e^2/a_B$ (≈ 27.2 eV)
- 1 unit of time $\leftrightarrow a_B/\alpha c$

etc.

Let's see how this works in the case of the hydrogen atom. In atomic units, the energy eigenvalues are

$$E_n = -\frac{1}{2n^2}.$$

In particular, the ground state energy is $-\frac{1}{2}$. For me, this is much easier to remember than the full expression. Then if I want to know the energy in, say, eV, I just remember that the fundamental unit of energy corresponds to 27.2 eV, so that $E_1 = -\frac{1}{2} \times 27.2 = 13.6$ eV. If you want to restore the full expression for the energy eigenvalues, just start multiplying by the factor of one that looks like an energy:

$$E_n = -\frac{1}{2n^2} = -\frac{1}{2n^2} \times \frac{e^2}{a_B}.$$

But also $(\alpha c)^2 m_e$ has units of energy, so we can write

$$E_n = -\frac{1}{2n^2} = -\frac{1}{2n^2} \times (\alpha c)^2 m_e.$$

The two expressions are of course equivalent, as you can check by plugging in the definitions of a_B and α. The hydrogen wavefunctions also simplify. The ground state, for example, is

$$\psi_{1s}(\vec{r}) = \frac{1}{\sqrt{\pi}} e^{-r}.$$

It would be good if you use atomic units on the upcoming problem sets to familiarize yourselves with them.

3) Eikonal approxmation

Jim Morehead promises that this week or next week you are going to spend some time on detailed Eikonal analysis/geometric optics in 210B, including looking at how the polarization vectors change along trajectories according to a sort of parallel transport. Here I'll try to work out some of the classical physics that has been bugging people and to relate the Hamilton-Jacobi equation of this problem to forms familiar from 210A.

Some classical physics: We can regard the action $S = \int_{t_0}^{t} L\, dt'$ of a physical trajectory as a function of coordinates and time at the upper limit of integration. Then

$$L = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}_i,$$

so that

$$\frac{\partial S}{\partial t} = -H(q_i, p_i, t).$$

Using $p_i = \frac{\partial S}{\partial q_i}$,

$$\frac{\partial S}{\partial t} + H(q_i, \frac{\partial S}{\partial q_i}, t) = 0, \qquad (1)$$

which is the Hamilton-Jacobi equation, a partial differential equation for S which is just a convenient way to rewrite Lagrangian or Hamiltonian mechanics.

A solution to this equation will be of the form

$$S = f(q_i, a_i, t) + A, \qquad (2)$$

depending on N constants a_i (in N degrees of freedom q_i) and a clock-setting constant A. We can consider f as a generating function for a canonical transformation from the old variables (q_i, p_i) to new variables $(b_i := \frac{\partial f}{\partial a_i}, a_i)$. (Recall that a function $F(q_i, P_i)$ which generates a canonical transformation from (q_i, p_i) to (Q_i, P_i) satisfies

$$dF = \sum_i p_i dq_i + \sum_i Q_i dP_i + (H' - H)dt, \tag{3}$$

where H' is the new Hamiltonian.) Since our f satisfies the Hamilton-Jacobi equation, the new Hamilton $H' = H + \frac{\partial f}{\partial t} = 0$, so that the equations of motion for the new coordinates are

$$\dot{a}_i = \dot{b}_i = 0.$$

The a_i and b_i are constants. Thus the program: We find the solution S to the Hamilton-Jacobi equation; we then form the equations $\frac{\partial S}{\partial a_i} = b_i$, which can be reexpressed as equations for the coordinates q_i in terms of the constants a_i, b_i.

For time-independent problems, the Hamiltonian will be constant along trajectories so that $S = \int L\, dt' = \int_{t_0}^{t} dt' (\sum_i p'_i dq'_i - E) := \widetilde{S}(q_i) - E(t - t_0)$. In this case the Hamilton-Jacobi equation reads

$$H(q_i, \frac{\partial \widetilde{S}}{\partial q_i}) = E. \tag{4}$$

a)

Let's make this concrete. The problem asks us to consider one component of the electromagnetic potential which has wave equation

$$\Big(\frac{n^2(\vec{x})}{c^2}\frac{d^2}{dt^2} - \nabla^2\Big)A^0(\vec{x}, t) = 0.$$

If the index of refraction $n(\vec{x})$ were independent of $\vec{x}$, we would have the usual plane wave solutions,

$$A^0 \sim e^{i\vec{k}\cdot\vec{x} - i\omega t}.$$

For $n(\vec{x})$ a slowly varying function of $\vec{x}$, the solutions will still look approximately like a plane wave, so we take a trial solution of the form $A^0 = e^{iS(\vec{x},t)/\hbar}$. As far as classical physics goes, $\hbar$ is just an expansion parameter, and like the WKB expansion, small $\hbar$ really means large S $(\sim kx)$. More carefully, if $\nabla n \sim 1/L$ defines the scale of change in n, we expand in large kL or small $1/kL$. The wavelength of the light should be small compared to the scale over which the wavelength changes appreciably.

Using the trial form for A^0, the wave equation becomes

$$i\hbar\frac{n^2(\vec{x})}{c^2}\frac{d^2S}{dt^2} - \frac{n^2(\vec{x})}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - i\hbar\nabla^2 S + (\nabla S)^2 = 0. \tag{5}$$

Expanding $S = S_0 + \hbar S_1 + O(\hbar^2)$, we have the equations

$$O(\hbar^0): \qquad -\tfrac{n^2(\vec{x})}{c^2}\left(\tfrac{\partial S_0}{\partial t}\right)^2 + (\nabla S_0)^2 = 0;$$

$$O(\hbar^1): \qquad i\tfrac{n^2(\vec{x})}{c^2}\tfrac{d^2S_0}{dt^2} - 2\tfrac{n^2(\vec{x})}{c^2}\tfrac{\partial S_0}{\partial t}\tfrac{\partial S_1}{\partial t} - i\nabla^2 S_0 + 2\nabla S_0 \cdot \nabla S_1 = 0.$$

If we rewrite the $O(\hbar^0)$ equation as

$$\frac{\partial S_0}{\partial t} = \frac{c}{n}|\nabla S_0|,$$

we have a Hamilton-Jacobi equation of the form (1) with

$$H(\vec{x}, \nabla S_0) = -\frac{c}{n(\vec{x})}|\nabla S_0|. \tag{6}$$

b)

Let's return to the wave equation (5). Since n is assumed to be independent of time, we can separate out the time dependence by the usual methods: Assume $S(\vec{x}, t) = S_x(\vec{x}) - S_t(t)$. When we vary t, holding $\vec{x}$ constant, $-i\hbar\nabla^2 S_x + (\nabla S_x)^2$ will not change, so neither can $-i\hbar\frac{n^2(\vec{x})}{c^2}\frac{d^2S_t}{dt^2} - \frac{n^2(\vec{x})}{c^2}\left(\frac{\partial S_t}{\partial t}\right)^2$. With a suggestive name for the separation constant $(-E^2)$, we must solve

$$-i\hbar\frac{d^2S_t}{dt^2} - \left(\frac{\partial S_t}{\partial t}\right)^2 = -E^2,$$

which has a solution $S_t = E(t - t_0)$. Thus our wave equation (5) is

$$-\frac{n^2(\vec{x})}{c^2}E^2 - i\hbar\nabla^2 S_x + (\nabla S_x)^2 = 0.$$

Re-expanding $S_x = S_0 + \hbar S_1 + O(\hbar^2)$, we have the equations

$$\begin{aligned} O(\hbar^0): &\qquad -\tfrac{n^2(\vec{x})}{c^2}E^2 + (\nabla S_0)^2 = 0; \\ O(\hbar^1): &\qquad -i\nabla^2 S_0 + 2\nabla S_0 \cdot \nabla S_1 = 0. \end{aligned} \tag{7}$$

I remind you that I have renamed S_0, S_1, so that now

$$S(\vec{x},t) = S_0(\vec{x}) + \hbar S_1(\vec{x}) - E(t-t_0) + O(\hbar^2).$$

As expected, the $O(\hbar^0)$ equation is a time-independent Hamilton-Jacobi equation (4):

$$H(\vec{x},\nabla S_0) = -\frac{c}{n(\vec{x})}|\nabla S_0| = E.$$

From now on I will consider only the $O(\hbar^0)$ term; the $O(\hbar^1)$ can be handled exactly as in the WKB case in the lecture notes. Looking at the $O(\hbar^0)$ equation (7), it is clear that if $n(\vec{x})$ is also independent of y (here a key assumption), the y dependence separates exactly as the time-dependence did. We let $S_0 = \widetilde{S}_0 + S_y(y)$, and again with a suggestive name for the separation constant (${p_y}_0^2$—the nought subscript will hopefully help make things clear later),

$$\left(\frac{\partial S_y}{\partial y}\right)^2 = {p_y}_0^2 \implies S_y = {p_y}_0(y-y_0).$$

Time for taking stock: We have expanded and reexpanded and separated and separated, finally arriving at

$$S(\vec{x},t) = \widetilde{S}_0(x) + {p_y}_0(y-y_0) - E(t-t_0) + O(\hbar). \tag{8}$$

The $O(\hbar^0)$ piece of the wave equation, which is our Hamilton-Jacobi equation, now reads

$$-\frac{n^2(x)}{c^2}E^2 + {p_y}_0^2 + \left(\frac{\partial \widetilde{S}_0}{\partial x}\right)^2 = 0.$$

Trivially then

$$\widetilde{S}_0 = \pm \int_{x_0}^{x} dx \sqrt{\frac{n^2(x)}{c^2}E^2 - {p_y}_0^2}$$

I hope you will forgive the slow-witted path we've taken to arrive at these results. Time- and y-translation invariance tells us that the potential $A^0 \sim e^{ip_y y - iEt}$, so we could have immediately jumped to the form (8). But since we are trying to understand methods for solving Hamilton-Jacobi problems, I went through the whole separation-of-variables plod. The two ways of thinking are of course equivalent.

c)

We have found our solution to the Hamilton-Jacobi equation of the form (2):

$$S = \pm \int_{x_0}^{x} dx \sqrt{\frac{n^2(x)}{c^2}E^2 - p_{y0}^2} + p_{y0}(y - y_0) - E(t - t_0) + O(\hbar). \qquad (9)$$

We can take the constants $a_i = (x_0, p_{y0})$ and the clock-setter $A = Et_0$. As discussed above, we consider (9) as a generating function for a canonical transformation where (x, y) are the old coordinates and the a_i the new 'momenta' (put in quotes because they are mixed coordinates and momenta). Then the new 'coordinates' are $b_i = \frac{\partial S}{\partial a_i}$. For instance,

$$b_1 = \frac{\partial S}{\partial p_{y0}} = y \mp \int_{x_0}^{x} dx \frac{p_{y0}}{\sqrt{\frac{n^2(x)}{c^2}E^2 - p_{y0}^2}}.$$

Since the new Hamiltonian $H' = 0$, $\dot{b}_1 = 0$, so that b_1 is a constant; call it y_0. (I think I called the constant 0 in section.) Then

$$y = y_0 \pm \int_{x_0}^{x} dx \frac{p_{y0}}{\sqrt{\frac{n^2(x)}{c^2}E^2 - p_{y0}^2}}. \qquad (10)$$

One way to find an expression for time is as follows: We have made a canonical transformation from phase space variables (x, p_x, y, p_y) to new phase space variables $(p_{x0}, x_0, -y_0, p_{y0})$. The latter are all constants in the sense that $H' = 0$. The former evolve according to the original

$$H = -\frac{c}{n}|\nabla S_0| = -\frac{c}{n}\sqrt{p_x^2 + p_y^2},$$

so that

$$\dot{y} = \frac{\partial H}{\partial p_y} = -p_y \frac{c}{n} \frac{1}{\sqrt{p_x^2 + p_y^2}} = \frac{p_y}{E}\frac{c^2}{n^2}.$$

From the generating-function formalism, we also know $p_y \equiv \frac{\partial S}{\partial y} = p_{y0}$. (As expected, $p_y = p_{y0}$, but they appear in this formalism as distinct canonical variables so I've given them different names.) Putting these results together,

$$y(t) = \frac{p_{y0}}{E}\frac{c^2}{n^2}t + y_0,$$

and plugging back into equation (10) gives

$$t = \int_{x_0}^{x} dx \, \frac{\frac{n^2(x)}{c^2} E}{\sqrt{\frac{n^2(x)}{c^2} E^2 - p_{y0}^2}}. \tag{11}$$

Alternately, note that the differential for the generating function (3) can be Legendre-transformed,

$$dS = p_x dx + p_y dy + p_{x0} dx_0 - y_0 dp_{y0} - H dt$$
$$\Longrightarrow$$
$$d\widetilde{S}_0 = d(S - p_{y0}(y - y_0) + H(t - t_0)) = p_x dx - y dp_y + p_{x0} dx_0 + p_{y0} dy_0 + (t - t_0) dH.$$

Then

$$t - t_0 = \left(\frac{\partial \widetilde{S}_0}{\partial H}\right) \qquad (\text{fixed } x, p_y = p_{y0}, x_0, y_0),$$

which gives the same result as (11).

At base all this analysis is governed by a simple principle which you should recognize from 210A: Hamiltonian evolution is canonical. In other words, the action, as a function of coordinates and momenta at the limits of integration, generates a canonical transformation between these variables, i.e. a canonical transformation from (x, p_x, y, p_y) to $(x_0, p_{x0}, y_0, p_{y0})$.

d)

We specialize to the case

$$n(x) = \begin{cases} n_1 & x < 0 \\ n_2 & x > 0, \end{cases}$$

and show Snell's Law: Looking on either side of the $x = 0$ plane, we have angles of incidence given by

$$\tan\theta_1 = \frac{-(y(x) - y(0))}{-x}; \qquad \tan\theta_2 = \frac{(y(x) - y(0))}{x},$$

for $x < 0$ and $x > 0$ respectively. (I am imagining the light ray moving on a trajectory increasing in both x and y.) Using the expression (10) for y with $y(0) = y_0 = 0$, the angles are

$$\tan\theta_1 = \frac{1}{x} \int_0^x dx \, \frac{p_y}{\sqrt{\frac{n_1^2}{c^2} E^2 - p_y^2}}; \qquad \tan\theta_2 = \frac{1}{x} \int_0^x dx \, \frac{p_y}{\sqrt{\frac{n_2^2}{c^2} E^2 - p_y^2}}.$$

The integrals are trivial, and we find

$$n_1 \sin\theta_1 = \frac{p_y}{E/c} = n_2 \sin\theta_2.$$