HW #3 (221B), due Feb 9, 4pm

1. For the hard sphere scattering problem

$$V = \begin{cases} \infty & (r < a) \\ 0 & (r > a) \end{cases}, \tag{1}$$

answer the following questions.

   (a) Show that

$$e^{2i\delta_l} = \frac{h_l^{(-)}(ka)}{h_l^{(+)}(ka)}. \tag{2}$$

   (b) Work out the leading low-k behavior of the phase shifts δ_l.

   (c) At large $k \gg a^{-1}$, study the behavior of partial wave cross sections $\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$ at large l and show that they quickly vanish beyond $l > ka$. Note that Mathematica does not have spherical Bessel functions built-in. They are related to the Bessel functions by

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z), \qquad n_l(z) = -\sqrt{\frac{\pi}{2z}} N_{l+1/2}(z) \tag{3}$$

   in my convention, and $J_n(z)$ is `BesselJ[n,z]`, while $N_n(z)$ is `BesselY[n,z]`.

   (d) Explain the above behavior of the partial wave cross sections analytically.

   (e) Based on the above results, show that $\sigma = 2\pi a^2$ by summing over l at high k and approximate the summation by integral over l.

2. Consider the "Delta-Shell" potential

$$V(r) = \gamma\delta(r-a) \tag{4}$$

and the scattering problem for the S-wave. Answer the following questions.

   (a) In the limit $\gamma \to \infty$, the regions inside ($r < a$) and outside ($r > a$) the shell decouple. What are the values of k for the states confined inside the shell?

   (b) Discuss when a bound state exists around $r = a$.

   (c) Show that the phase shift is given by

$$e^{2i\delta_0} = \frac{1 + \frac{2m\gamma}{\hbar^2 k} e^{-ika}\sin ka}{1 + \frac{2m\gamma}{\hbar^2 k} e^{ika}\sin ka} = e^{-2ika}\frac{\sin ka + \frac{\hbar^2 k}{2m\gamma} e^{ika}}{\sin ka + \frac{\hbar^2 k}{2m\gamma} e^{-ika}}. \tag{5}$$

   Verify that $\gamma \to \infty$ gives the hard sphere, while $\gamma \to 0$ no scattering.