HW #2 Solutions (221B)

1) See mathematica program

2) Yukawa potential in the Born approximation

a)

By straight forward integration in the Born approximation (see e.g. lecture notes 2),

$$\begin{aligned}
f^{(1)}(\vec{k}, \vec{k}') &= -\tfrac{1}{4\pi}\tfrac{2m}{\hbar^2}\int d\vec{x}\, V(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \\
&= -\tfrac{2m}{\hbar^2}\tfrac{1}{q}\int_0^\infty dr\, r^2 V_0 \tfrac{e^{-r/a}}{r}\sin qr \\
&= -\tfrac{2mV_0}{\hbar^2}\tfrac{1}{q}\left(\tfrac{e^{iqr-r/a}}{2i(iq-1/a)} - \tfrac{e^{-iqr-r/a}}{2i(-iq-1/a)}\right)_0^\infty \\
&= -\tfrac{2mV_0}{\hbar^2}\tfrac{1}{q^2+1/a^2},
\end{aligned}$$

where $q^2 = |\vec{k} - \vec{k}'|^2 = 2k^2(1-\cos\theta)$. Therefore,

$$\begin{aligned}
\sigma = \int \tfrac{d\sigma}{d\Omega} d\Omega &= \int 2\pi \left(\tfrac{mV_0}{\hbar^2 k^2}\right)^2 \tfrac{d\cos\theta}{(1-\cos\theta+1/2k^2a^2)^2} \\
&= 2\pi \left(\tfrac{mV_0}{\hbar^2 k^2}\right)^2 \tfrac{1}{1-\cos\theta+1/2k^2a^2}\Big|_{-1}^{1} \\
&= 4\pi a^2 \, \tfrac{m^2 V_0^2 a^2}{\hbar^4} \tfrac{4}{1+4k^2a^2}.
\end{aligned}$$

b)

As discussed in the lecture notes, we require that the difference between the true wavefunction ψ and the free plane wave ϕ be small where the potential is large. We compute in the Born approximation at the origin (relabling $\vec{x}' \to \vec{x}$):

$$|\psi - \phi| \sim \frac{2m}{\hbar^2}\left|\int d\vec{x}\, \frac{e^{ikr}}{4\pi r}\frac{V_0\, e^{-r/a}}{r} e^{ikz}\right| \ll 1.$$

The integral is much easier to do if you integrate the r variable first.

$$\frac{2m}{\hbar^2}\int 2\pi\, d\cos\theta\, r^2 dr \frac{e^{ikr}}{4\pi r}\frac{V_0\, e^{-r/a}}{r} e^{ikr\cos\theta} = \frac{mV_0}{\hbar^2}\int d\cos\theta \frac{-1}{ik + ik\cos\theta - 1/a}$$

$$= -\frac{mV_0}{ik\hbar^2} \log\left(\cos\theta + 1 - 1/ika\right)\Big|_{-1}^{1} = -\frac{mV_0}{ik\hbar^2} \log\left(1 - 2ika\right).$$

If you integrate θ first, you can then do the r integral in Mathematica and use the relation $\arctan z = \frac{\log 1+iz}{\log 1-iz}$ from Gradshtein and Ryzhik to turn the result into the single log as above. The condition for the validity of the Born approximation is

$$\frac{mV_0}{k\hbar^2} \left|\log\left(1 - 2ika\right)\right| \ll 1$$

c)

Given the above condition, we also have

$$\frac{m^2V_0^2}{k^2\hbar^4} \left|\log\left(1 - 2ika\right)\right|^2 \ll 1.$$

Though this weakens the meaning of $\ll$, the new condition will be sufficient for our purposes. Write the new condition as

$$\gamma\, v(k) \ll 1,$$

where

$$\gamma = \frac{m^2V_0^2a^2}{\hbar^4} \quad \text{and} \quad v(k) = \frac{\left|\log\left(1 - 2ika\right)\right|^2}{k^2a^2}.$$

In this language,

$$\sigma = 4\pi a^2\, \gamma\, s(k),$$

where

$$s(k) = \frac{4}{1 + 4k^2a^2}.$$

If we can show $v(k) \geq s(k)$, then the validity condition $\gamma v(k) \ll 1$ implies $\sigma \ll 4\pi a^2$ whenever the Born approximation is valid. I plotted the two functions $v(k)$ and $s(k)$ in mathematica and displayed the result in "yukawa.nb" at the end of this solution. The plot shows that $v(k)$ is in fact an upper bound for $s(k)$, and this holds independent of a (though I only plotted one case, $a = 1$.

We can arrive at this upper bound analytically as follows. First find the magnitude of $\log(1-2ika)$ by writing $z = 1-2ika = re^{i\theta}$, taking the logarithm of $re^{i\theta}$ (principal branch), and then taking the magnitude of the resulting complex number:

$$\log(1+2ika) = \log\sqrt{1+4k^2a^2}\, e^{i\arctan(-2ka)} = \log\sqrt{1+4k^2a^2} + i\arctan(-2ka),$$

$$\text{s.t.}\ \ |\log(1-2ika)| = \left(\frac{1}{4}\log^2(1+4k^2a^2) + \arctan^2(-2ka)\right)^{1/2}.$$

Since $v(k)$ and $s(k)$ are strictly real and positive for real k, the condition $v(k) \geq s(k)$ is equivalent to $v(k)/s(k) \geq 1$, or

$$f(k) := \frac{v(k)}{s(k)} = \left(\frac{1}{4}\log^2(1+4k^2a^2) + \arctan^2(-2ka)\right)\frac{1+4k^2a^2}{4k^2a^2} \geq 1.$$

Now, $\lim_{k\to 0} f(k) = 1$, the $\arctan^2(-2ka)$ contributing the $4k^2a^2$ needed to balance the similar factor in the denominator. So initially $f(0) = 1 \geq 1$. By taking derivatives with respect to k (e.g. on Mathematica), you can confirm that $f'(k) \geq 0$ for k real and positive, so that f is strictly increasing. Thus $f(k) = \frac{v(k)}{s(k)} \geq 1$.

If you do this problem by making approximations for various values of k, your results will look slightly different from those in the lecture notes. Prof. Murayama's definition of $V = V_0 \frac{e^{-r/a}}{r}$ in the problem means V_0 has dimension $energy * length$ as opposed to dimensions of $energy$ as for the V_0 in the lecture notes. To recover the results from the lecture notes redefine V_0 in this problem $V_0 \to V_0\, a$.

In[20]:= $s[k\_] := \frac{4}{1 + 4\, k^2\, a^2}$

In[19]:= $v[k\_] := \frac{Abs[Log[1 - 2\, I\, a\, k]]^2}{k^2\, a^2}$

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Plot[{v[k] /. a → 1, s[k] /. a → 1}, {k, 0, 10}, PlotRange → {0, 4},
  PlotStyle → {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}]
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