HW #1 (221B), due Jan 26, 4pm

1. Show that Lippmann–Schwinger equation in one dimension is given by
$$\psi(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-mi}{\hbar^2 k} \int dx' e^{ik|x-x'|} V(x') \psi(x'). \tag{1}$$
$k > 0$ is assumed. The first term corresponds to the incident particle, while the second term the scattered wave.

2. For large $r = |x| \gg a$, and assuming that the scattering potential $V(x')$ is sizable only in a small region $|x'| \lesssim a$, write down the asymptotic form of the wave function. Use $|x - x'| = \sqrt{(x - x')^2} = r - x'x/r + O(x')^2$, and rewrite the wave function in the form
$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \left[e^{ikx} + f(k', k) e^{ikr} \right], \tag{2}$$
and work out what $f(k', k)$ is. Compared to the three-dimensional case, the function $f(k', k)$ takes only two values, $k' = \pm k$, because of only one dimension.

3. Now consider the potential $V(x) = \gamma\delta(x)$ and answer the following questions.
   (a) Write down the wave function $\psi(x)$ from the exact form Eq. (1).
   (b) Verify that the obtained $\psi(x)$ indeed satisfies the Schrödinger equation. In order to do so, you need to know what the delta function potential does. Schrödinger equation is the same as the free equation for $x \neq 0$. By integrating the Schrödinger equation from $x = -\epsilon$ to $x = \epsilon$, and taking the limit of $\epsilon \to 0$, you find that the derivative of the wave function is discontinuous at $x = 0$, *i.e.*, $\psi'(+0) - \psi'(-0) \propto \gamma$ where $\psi' = d\psi/dx$.
   (c) Show that there is a pole in the complex k plane in the scattered wave, which corresponds to a bound state solution (exponentially decaying solution at both infinities) if and only if $\gamma < 0$. Write down the bound state wave function.
   (d) Show, however, a delta function potential $\gamma\delta(\vec{x})$ does not lead to any scattering in three dimensions, again using Lippmann-Schwinger equation.