

# 221B Lecture Notes

## Quantum ElectroDynamics

### 1 Putting Everything Together

Now we are in the position to discuss a truly relativistic, quantum formulation of electrodynamics. We have discussed all individual elements of this already, and we only need to put them together. But putting them together turns out to give us new phenomena, and we will briefly discuss them.

To discuss electrons, we need the Dirac field, not a single-particle Dirac equation as emphasized in the previous lecture note. To discuss the radiation field and photons, we need quantized radiation field. We know how they couple to each other because of the gauge invariance. In the end, the complete action for the Quantum ElectroDynamics (QED) is

$$S = \int dt d\vec{x} \left[ \psi^\dagger \left( i\hbar \frac{\partial}{\partial t} - e\phi - c\vec{\alpha} \cdot (-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}) - mc^2\beta \right) \psi + \frac{1}{8\pi} (\vec{E}^2 - \vec{B}^2) \right]. \quad (1)$$

This is it!

Now the problem is develop a consistent perturbation theory. It is done in the following way. You separate the action into two pieces,  $S_0$  and  $S_{int}$ ,

$$S_0 = \int dt d\vec{x} \left[ \psi^\dagger \left( i\hbar \frac{\partial}{\partial t} + ic\vec{\alpha} \cdot \hbar\vec{\nabla} - mc^2\beta \right) \psi + \frac{1}{8\pi} (\vec{E}^2 - \vec{B}^2) \right], \quad (2)$$

and

$$S_{int} = \int dt d\vec{x} \psi^\dagger (-e\phi + e\vec{\alpha} \cdot \vec{A}) \psi. \quad (3)$$

The unperturbed action  $S_0$  consists of that of free Dirac field and free radiation field. We've done this before, and we know what we get: Fock space for spin 1/2 electrons and spin one photons. The interaction part is what is proportional to  $e$ . We regard this piece as the perturbation. Therefore, perturbation theory is a systematic expansion in  $e$ . It turns out that the expansion is actually in a dimensionless quantity

$$\frac{1}{\pi} \frac{e^2}{\hbar c} = \frac{\alpha}{\pi} = \frac{1}{430} \ll 1 \quad (4)$$

and the perturbation theory is extremely well-behaved.

The actual calculations are done using Feynman diagrams. Feynman's ingenious invention is to organize perturbation series in terms of graphs with a well-defined rule how a graph is related to a quantum mechanical amplitude quantitatively. We will not develop such a theory in this course; you can study it in 229A if you are interested. Instead, we discuss a few examples which appear as a result of full quantum effects both in matter and radiation.

## 1.1 Lamb Shift

We learned with Dirac equation that states of hydrogen atom with the same principal quantum number  $n$  and the total angular momentum  $j$  remain degenerate despite the corrections from spin-orbit coupling, relativistic corrections, and Darwin term. They are, however, split as a result of full quantum interactions between the electron and photons. This is what Willis Lamb found after he worked on war-time radar technology during the WWII and came back to his lab, applied his radar technology to the hydrogen atom. He found transition spectrum between  $2s_{1/2}$  and  $2p_{1/2}$  states at about 1 GHz.

You heard about the Darwin term pushing the  $s$ -states up because the Zitterbewegung smears the electric field and gives rise to a delta-function potential at the origin. If there is an additional reason for the jitter of the electron, it would contribute more to the similar effect. The additional reason is the zero-point fluctuation of the radiation field. Each momentum mode of the photon has the zero-point fluctuation, and each of them jiggles the electron. That would make the electron jitter a little bit more in addition to the Zitterbewegung and pushes the  $s$ -state further up.

Let us for simplicity treat the electron non-relativistically and see how much it gets jiggled by the zero-point motion of the electric field. The classical equation of motion for the “jiggle” part of the electron position is

$$\delta\ddot{\vec{x}} = \frac{e}{m}\vec{E}. \quad (5)$$

As we discussed in the case of the Darwin term generated by the Zitterbewegung, such a “jiggle” would smear the electric field and generate additional potential term

$$\Delta V = \frac{1}{2}\langle\delta x^i\delta x^j\rangle\frac{\partial^2 e\phi}{\partial x^i\partial x^j} = \frac{1}{6}\langle(\delta\vec{x})^2\rangle\Delta(e\phi). \quad (6)$$

We are interested in the electric field caused by the zero-point motion. For each frequency mode of the photon, with frequency  $\omega$ , we find then

$$\delta\vec{x}_\omega = -\frac{e}{m\omega^2}\vec{E}_\omega. \quad (7)$$

Therefore the size of the fluctuation in the electron position is

$$\langle(\delta\vec{x}_\omega)^2\rangle = \frac{e^2}{m^2\omega^4}\langle\vec{E}_\omega^2\rangle. \quad (8)$$

For each frequency mode, the zero-point energy of the photon is

$$\int d\vec{x} \frac{1}{8\pi}(\vec{E}_\omega^2 + \vec{B}_\omega^2) = \frac{1}{2}\hbar\omega. \quad (9)$$

Because  $\vec{E}^2 = \vec{B}^2$  for a photon, we find

$$\langle\vec{E}_\omega^2\rangle = \frac{2\pi}{L^3}\hbar\omega. \quad (10)$$

We now sum the contribution of each momentum modes incoherently (because each of them gives a random jiggling in an uncorrelated way) for both polarization states, and find

$$\begin{aligned} \langle(\delta\vec{x})^2\rangle &= 2L^3 \int \frac{d\vec{k}}{(2\pi)^3} \langle(\delta\vec{x}_\omega)^2\rangle \\ &= 2L^3 \int \frac{4\pi k^2 dk}{(2\pi)^3} \frac{e^2}{m^2\omega^4} \frac{2\pi}{L^3} \hbar\omega \\ &= \frac{2}{\pi} \frac{e^2}{m^2} \int \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{\omega^4} \hbar\omega \\ &= \frac{2\hbar e^2}{\pi m^2 c^3} \int \frac{d\omega}{\omega}. \end{aligned} \quad (11)$$

The integral over  $\omega$  is logarithmically divergent both in the ultraviolet ( $\omega \rightarrow \infty$ ) and the infrared ( $\omega \rightarrow 0$ ).

Our calculation so far is clearly not valid when the electron is jiggled by a photon of  $\omega$  larger than  $mc^2/\hbar$  because the electron would become relativistic. Therefore, we assume that the correct fully relativistic treatment will cut off the integral around  $\omega \sim mc^2/\hbar$ . Indeed, calculations using Feynman diagrams show that this is indeed the case. Because the integral depends

only logarithmically on the cutoff, it should not make a big difference if the cutoff is actually  $2mc^2/\hbar$  or  $mc^2/2\hbar$ , etc. We take it to be simply  $mc^2/\hbar$ . In the infrared, frequency smaller than the time for the electron to move inside the atom should not affect the result because the “jiggle” would be overshadowed by the motion of the electron itself. Therefore the integral is cutoff at around  $\omega \sim cZa_B^{-1} = mcZe^2/\hbar^2$ . As a result, the size of the “jiggling” is approximately

$$\langle(\delta\vec{x})^2\rangle \simeq \frac{2\hbar e^2}{\pi m^2 c^3} \log \frac{mc^2/\hbar}{mcZe^2/\hbar^2} = \frac{2\hbar^2\alpha}{\pi m^2 c^2} \log \frac{1}{Z\alpha}, \quad (12)$$

with  $\alpha = e^2/\hbar c = 1/137$ . Following the calculation of the energy shift of  $s$ -states from the Darwin term, we find the additional potential term Eq. (6) to be

$$\begin{aligned} \Delta V &= \frac{1}{6} \langle(\delta\vec{x})^2\rangle \Delta(e\phi) \\ &\simeq \frac{1}{6} \frac{2\hbar^2\alpha}{\pi m^2 c^2} \log \frac{1}{Z\alpha} 4\pi Z e^2 \delta(\vec{x}) \\ &= \frac{4\hbar^3 Z \alpha^2}{3m^2 c} \log \frac{1}{Z\alpha} \delta(\vec{x}). \end{aligned} \quad (13)$$

The resulting energy shift for  $nS$ -states is

$$\begin{aligned} \Delta E_n &\simeq \frac{4\hbar^3 Z \alpha^2}{3m^2 c} \log \frac{1}{Z\alpha} |\psi_n(0)|^2 \\ &= \left[ \frac{8(Z\alpha)^4 \alpha}{3\pi} \log \frac{1}{Z\alpha} \right] mc^2 \frac{1}{2n^3}. \end{aligned} \quad (14)$$

Therefore, this contribution is suppressed relative to the fine-structure by  $\alpha/\pi$  but is enhanced by a logarithm  $\log \alpha^{-1} = 4.9$ . For  $n = 2$ , it gives about 1 GHz for the microwave resonance frequency between  $2p$  and  $2s$ , in rough agreement with data as we will see below.

The standard calculation uses Feynman diagrams, where the electron emits a virtual photon before it interacts with the Coulomb potential, and after the interaction it reabsorbs the virtual photon. This diagram, called the vertex correction, is actually divergent both in the ultraviolet and the infrared; reminiscent of the discussion above. It turns out, however, that the ultraviolet divergence is of a different character. The piece that corresponds to the amount of “jiggling” of the electron, more correctly called the “charge

radius” of the electron, is actually ultraviolet finite in the fully relativistic calculations, supporting the rough “cutoff” at  $\omega \sim mc^2/\hbar$  employed above. The ultraviolet divergence in this Feynman diagram is properly cancelled by another ultraviolet divergence called “wave function renormalization.” When you use the second-order perturbation theory, your state  $|\phi_n\rangle$  is modified to

$$|\phi_n\rangle \rightarrow |\chi_n\rangle = |\phi_n\rangle + \sum_{i \neq n} \frac{|i\rangle \langle i|V|0\rangle}{E_0 - E_i} + \sum_{i,j \neq n} \frac{|j\rangle \langle j|V|i\rangle \langle i|V|0\rangle}{(E_n - E_j)(E_n - E_i)}, \quad (15)$$

where  $|0\rangle$  is the zeroth order state (not the vacuum) and  $|i\rangle$  other states that mix with  $|0\rangle$  due to the perturbation  $V$ . However, this state is not correctly normalized, because

$$\langle \chi_n | \chi_n \rangle = 1 + \sum_i \frac{|\langle \phi_n | V | i \rangle|^2}{(E_n - E_i)^2}. \quad (16)$$

To correctly normalize the perturbed state  $|\chi_n\rangle$ , we need to “renormalize” it as

$$|\chi_n\rangle' = |\chi_n\rangle \frac{1}{1 + \frac{1}{2} \sum_i |\langle \phi_n | V | i \rangle|^2 / (E_n - E_i)^2} \quad (17)$$

In the case of the QED, it corresponds to the Feynman diagram where the electron emits a virtual photon and reabsorbs it without any other interactions, which is also ultraviolet divergent. This change in the normalization of the state can be shown to precisely cancel the ultraviolet divergence in the vertex correction, and hence there is no problem with the apparent divergences.

The present theoretical and experimental situation is reviewed, for example, in M.I. Eides, H. Grotch and V.A. Shelyuto, “Theory of light hydrogen-like atoms,” *Phys. Rep.*, **342**, 63-261, (2001). The best experimental value of the  $2s$ – $2p$  splitting is

$$1.057\,845(3) \text{ GHz}. \quad (18)$$

The theoretical calculations depend on variety of other corrections in addition to the effect I had discussed, including the fact that the charge of the proton is not strictly point-like. The charged radius is not well determined experimentally, and limits the theoretical accuracy in calculating the level splitting. Using one particular measurement of the proton charge radius 0.862(12) fm, the theory gives

$$1.057\,833(2)(4) \text{ GHz}, \quad (19)$$

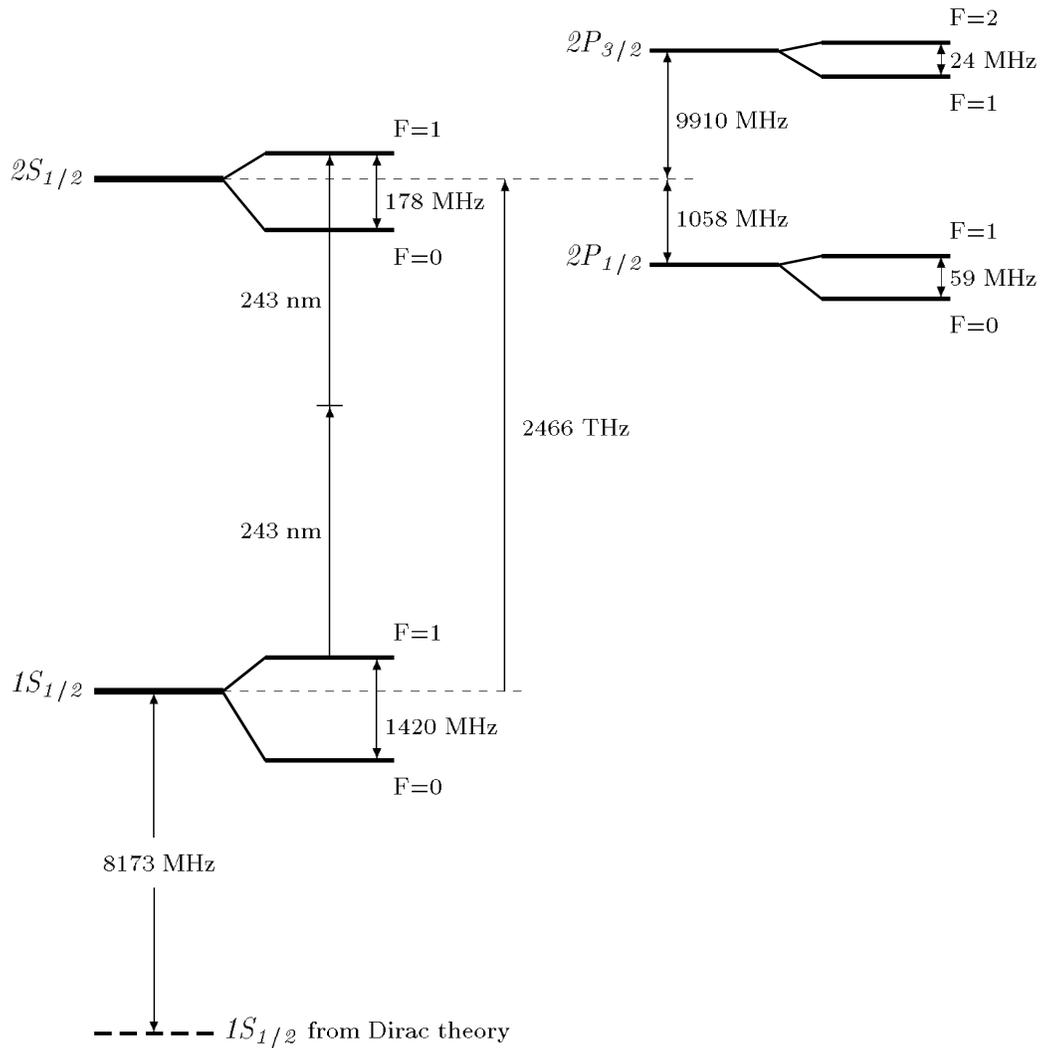


Figure 1: Low-lying hydrogen atomic levels with Lamb shifts and hyperfine splittings. Taken from M.I. Eides, H. Grotch and V.A. Shelyuto, "Theory of light hydrogenlike atoms," *Phys. Rep.*, **342**, 63-261, (2001).

which disagrees with data at more than 2 sigma level. But other measurements of the proton charge radius disagree with this measurement, and the discrepancy becomes larger. The inconsistency among the data makes it impossible for us to draw any conclusions beyond a simple qualitative statement that theory and data agree for 6 digits.

## 1.2 Anomalous Magnetic Moment

We learned from the Dirac equation that the gyromagnetic ratio  $g = 2$ . This is certainly in a good agreement with data. But both experiments and theoretical calculations had progressed greatly since old days. (See V.W. Hughes and T. Kinoshita, *Rev. Mod. Phys.* **71**, S133-S139 (1999). )

The crucial theoretical progress usually attributed to Schwinger is that there is a correction at  $O(\alpha)$  to the  $g$ -factor. Because of the quantum fluctuations, an electron is sometimes not just an electron but is accompanied by a photon. More precisely, the process is the following. An electron interacts with the magnetic field with  $g = 2$ . But before the electron interacts with the magnetic field, it can emit a (virtual) photon, violating the energy conservation. The electron then interacts with the magnetic field, and only after that, it reabsorbs the photon. Schwinger's calculation showed that the  $g$ -factor is not longer precisely 2 but is corrected as

$$g = 2 \left( 1 + \frac{\alpha}{2\pi} \right). \quad (20)$$

More recently, Kinoshita has calculated the correction up to  $O(\alpha^4)$  with 891 Feynman diagrams. It is customary to quote  $a_e = (g_e - 2)/2$ . He found

$$\begin{aligned} a_e = & \frac{\alpha}{2\pi} - 0.328\,478\,965\dots \left(\frac{\alpha}{\pi}\right)^2 + 1.181\,241\,456\dots \left(\frac{\alpha}{\pi}\right)^3 \\ & - 1.509\,8(384) \left(\frac{\alpha}{\pi}\right)^4 + 4.393(27) \times 10^{-12}. \end{aligned} \quad (21)$$

The last constant is a contribution from known heavier particles, such as  $\mu$ ,  $\tau$  leptons, hadrons, and  $W$ ,  $Z$  bosons. In order to obtain a numerical value, we need to know the fine structure constant  $\alpha$  very precisely. The best measurement comes from the Quantum Hall Effect,

$$\alpha^{-1} = 137.036\,003\,7(33). \quad (22)$$

Using this value as the input, the QED predicts the electron magnetic moment to be

$$a_e = 1\,159\,652\,153.5(1.2)(28.0) \times 10^{-12}. \quad (23)$$

The first uncertainty comes from the estimated error in the numerical integrations involved in the calculations, while the latter (and the dominant) error is in the input value of  $\alpha$ .

The experiment also had made a dramatic progress. The most precise measurement was done using Penning trap. If  $g$  were exactly two, an electron moving in a magnetic field would have precisely the same values for the cyclotron frequency and the spin precession frequency. The difference between them measures  $g - 2$ . The best values due to Van Dyck are

$$a_{e^-} = 1\,159\,652\,188.4(4.3) \times 10^{-12}, \quad (24)$$

$$a_{e^+} = 1\,159\,652\,187.9(4.3) \times 10^{-12}. \quad (25)$$

The agreement between experiment and theory is truly amazing. The experimental accuracy is  $4 \times 10^{-12}$ , which agrees with theory at 1.3 sigma level. Incredible! This is *the* most dramatic success of quantum physics, most likely of all physical sciences.

The anomalous magnetic moment of muon is also interesting. Because muon is short-lived (lifetime is microsecond), the experimental measurement is more difficult. The trick is to actually use the decay product of the muon,  $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ , where you can detect the electron (but not neutrinos). Luckily, parity is violated in this decay (!), and the direction of the decay electron is correlated with the muon spin. By measuring the direction of the decay electrons, we can measure the muon spin precession and hence  $g_\mu - 2$ . Theoretically, muon is heavier and the anomalous magnetic moment is more sensitive to heavier particles than that of electron. In fact, the contribution from hadrons (pions, various mesons, protons, etc) is quite important. You may even hope that it may detect the effect of yet-undiscovered particles.

The theoretical prediction is

$$a_\mu = 11\,659\,159.6(6.7) \times 10^{-10}. \quad (26)$$

Currently a new experiment is being conducted at Brookhaven National Laboratory and has measured the (anti-)muon magnetic moment. They reported

$$a_\mu = 11\,659\,202(14)(6) \times 10^{-10}. \quad (27)$$

The data and theory do not agree at 2.6 sigma! The experimental group speculated that this may be due to the quantum effect of yet-undiscovered supersymmetric partner of muon,  $\tilde{\mu}$ . For more details, go to <http://phyppro1.phy.bnl.gov/g2muon/index.shtml>.

### 1.3 Vacuum Polarization

The Dirac sea is a collection of infinite number of electrons in negative energy states. Even though it has an infinite (negative) charge, as long as it is completely homogeneous, we will not be able to detect it because there is no preferred direction to produce an electric field. However, if the homogeneity is broken, for example by the presence of a point charge, the distribution of the negative energy electrons is no longer homogeneous and can have a physically observable effect.

Suppose you place a positive point charge in the “vacuum” filled with negative energy electrons. The negative energy electrons are then attracted to the positive point charge, effectively screening the point charge. Therefore, viewed from far away, the charge of the point charge would appear less than what you originally put in. As you go closer and closer to the point charge, you see less effect due to the screening by negative energy electrons, and hence the charge appears larger and larger. In other words, the fine-structure constant  $\alpha$  would appear larger and larger as you measure it at smaller and small distance scales. And smaller distance scales, because of the uncertainty principle, corresponds to the larger momentum transfer. This is the effect of the “vacuum polarization,” causing the fine-structure constant to grow at higher momentum transfers. We often say that the fine-structure constant *runs*.

This effect, even though theoretically well worked out within the QED, had been measured for the first time in 1997, by an experiment at the electron-positron collider TRISTAN (I. Levine *et al*, “Measurement of the Electromagnetic Coupling at Large Momentum Transfer,” *Phys. Rev. Lett.* **78**, 424427 (1997)). Compared to the value of the fine-structure constant Eq. (22) at zero momentum transfer (a negligible momentum transfer compared to  $mc$ ) such as in Quantum Hall Effect, they reported the fine-structure constant at momentum transfer  $Q^2 = (57.77 \text{ GeV}/c)^2$  to be  $\alpha^{-1}(Q^2) = 128.5 \pm 1.8 \pm 0.7$ , which agrees with the theoretical prediction  $129.6 \pm 0.1$ . Note that the vacuum polarization effect is not just due to the negative energy electrons, but also due to mu, tau leptons, and all hadrons (*i.e.*, quarks).

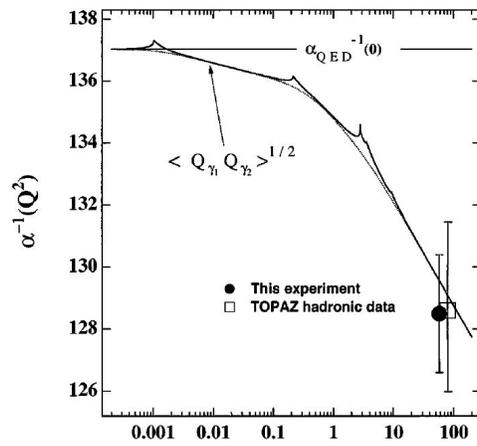


Figure 2: The measured and theoretical fine-structure constant as a function of momentum transfer  $Q$ . The solid and dotted lines correspond to positive and negative  $Q^2$  predictions, respectively. As we probe closer to the bare charge, its effective strength increases.  $\langle Q_{\gamma_1} Q_{\gamma_2} \rangle^{1/2}$  denotes the square root of the median value for the product of the photon momentum transfers in the anti-tagged  $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$  sample. The hadronic data point has been shifted for display. Taken from I. Levine *et al.*