221A Lecture Notes
Steepest Descent Method

1 Gamma Function

The best way to introduce the steepest descent method is to see an example. The Stirling's formula for the behavior of the factorial $n!$ for large n is

$$n! \sim \sqrt{2\pi n}\; n^n e^{-n} \tag{1}$$

which can be obtained from the integral representation of the Gamma function using the steepest descent method.

1.1 How to do it

The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty dt t^{x-1} e^{-t}. \tag{2}$$

For the integer arguments, the Gamma function becomes a factorial, $\Gamma(n) = (n-1)!$. Therefore we should study $\Gamma(n+1) = n!$

$$n! = \int_0^\infty dt t^n e^{-t}. \tag{3}$$

The point is that the integrand $t^n e^{-t}$ is peaked at some value of t (see below) and its form can be approximated by a Gaussian if n is large. To see this, let us first write it as

$$n! = \int_0^\infty dt e^{-t+n\ln t}. \tag{4}$$

The exponent $f(t) = -t+n\ln t$ is at its maximum where $f'(t) = -1+n/t = 0$, or $t = n$. We can expand the exponent around the maximum up to the second order

$$f(t) = -n+n\ln n+\frac{1}{2}f''(n)(t-n)^2+O(t-n)^3 = -n+n\ln n-\frac{1}{2n}(t-n)^2+O(t-n)^3. \tag{5}$$

Let us ignore the terms of $O(t-n)^3$ and above. We will come back to the question when it is a good approximation afterwards. Then the integral is given by

$$n! \simeq \int_0^\infty dt e^{-n+n\ln n-\frac{1}{2n}(t-n)^2}. \tag{6}$$

Because the Gaussian integral is around $t=n$ and the standard deviation $\sqrt{n}$, we can extend the integration region to $-\infty$ to ∞ with negligible error for large n. Then we find

$$n! \simeq e^{-n+n\ln n}\sqrt{2\pi n} = \sqrt{2\pi n}\; n^n e^{-n}, \tag{7}$$

which is nothing but the Stirling's formula Eq. (1).

1.2 Is it good?

To see if this is a good approximation, let us redo the calculation more carefully. Because we are expanding around $t=n$, change the variable to s by $t=n(1+s)$ in Eq. (4), and we find

$$n! = \int_{-1}^\infty e^{-n(1+s)+n\ln n(1+s)} n ds = n^{n+1}e^{-n}\int_{-1}^\infty e^{-n(s-\ln(1+s))}ds. \tag{8}$$

The exponent is expanded as

$$-n(s-\ln(1+s)) = -n\sum_{k=2}\frac{(-1)^k}{k}s^k. \tag{9}$$

So far everything is exact.

Now there are two approximations we make. One is to keep only the second power in s, and the other is to extend the integration region to the entire real axis.

The first approximation to keep only the second power in s in the exponent is justified as follows. The correction to this approximation is in the higher powers in s in Eq. (9). Assuming that the Gaussian integral is done, how big are these terms? Because the standard deviation in s is $1/\sqrt{n}$, the k-power term $(-1)^k s^k/k$ is estimated as roughly $1/n^{k/2}$, which is suppressed relative to the leading term $s^2 \sim 1/n$ for large n.

The second approximation is to extend the integration region. Again for large n, the Gaussian factor is already quite damped at $s=-1$ already as e^{-n}.

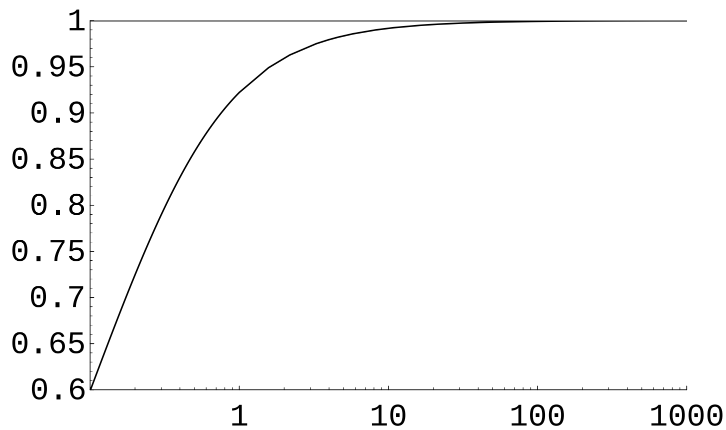

Figure 1: The ratio of the Stirling's formula Eq. (1) compared to the true value of the Gamma function $\Gamma(n+1)$, plotted against n.

Therefore extending the integration region induces only an exponentially suppressed error.

In all, both approximations are justified for large $n \gg 1$.

If you keep higher orders in $1/n$, you obtain an expansion in $1/n$ for $n!$, or more generally, an expansion in $1/x$ for $\Gamma(x)$. Up to tenth order,

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x} \left(1 + \frac{1}{12\,x} + \frac{1}{288\,x^2} - -\frac{139}{51840\,x^3} - \frac{571}{2488320\,x^4} + \frac{163879}{209018880\,x^5} + \frac{5246819}{75246796800\,x^6} - \frac{534703531}{902961561600\,x^7} - \frac{4483131259}{86684309913600\,x^8} + \frac{432261921612371}{514904800886784000\,x^9} + O(x^{-10})\right). \quad (10)$$

It should be noted that the expansion in $1/x$ for the Gamma function $\Gamma(x)$ is known as an *asymptotic series*, not a Taylor series. An asymptotic series in a parameter ϵ of a function is given in a power series

$$f(\epsilon) = \sum_{n=0}^{\infty} f_n \epsilon^n \quad (11)$$

where the series actually *does not converge*. Instead, if you truncate the series at an order N

$$f_N(\epsilon) = \sum_{n=0}^{N} f_n \epsilon^n, \quad (12)$$

Table 1: The numerical result of the asymptotic series for the Gamma function $\Gamma(x)$, for $x = 1$ and N up to 30. $N = 1$ corresponds to the Stirling's formula $\sqrt{2\pi}x^{x-1/2}e^{-x}$. The true value is $\Gamma(1) = 1$. Up to $N = 6$ the series keeps improving, while higher N makes the series less and less accurate and eventually diverge.

N	$f_N(1)$	N	$f_N(1)$	N	$f_N(1)$
1	0.922137	11	1.00053	21	-0.235955
2	0.998982	12	0.998768	22	12.1188
3	1.00218	13	0.998618	23	13.1524
4	0.999711	14	1.00452	24	-131.44
5	0.999499	15	1.00502	25	-143.527
6	1.00022	16	0.977792	26	1878.31
7	1.00029	17	0.975504	27	2047.24
8	0.999741	18	1.14106	28	-31242.9
9	0.999693	19	1.15495	29	-34023.3
10	1.00047	20	-0.128483	30	603493.

the difference between the true value $f(\epsilon)$ and and approximate expression $f_N(\epsilon)$ goes to zero $(f(\epsilon) - f_N(\epsilon))/\epsilon^N \to 0$ as $\epsilon \to 0$.

In the case of the Gamma function, the source of the non-convergence is pretty clear: you are ignoring corrections suppressed as e^{-n} when you extend the integration region to infinity. In an expansion in $\epsilon = 1/n$, corrections of order $e^{-n} = e^{-1/\epsilon}$ are zero to arbitrary high orders in ϵ. This is because any derivatives of $e^{-1/\epsilon}$ in ϵ are exponentially suppressed at $\epsilon = 0$, and cannot be Taylor expanded at $\epsilon = 0$.

2 Spherical Bessel Function

Here is another example. Spherical Bessel Function appears in solutions to the free Schrödinger equation in the spherical coordinate. We use the large l behavior of the spherical Bessel function in some scattering problems. We can obtain the behavior from the integral representation

$$j_l(\rho) = \frac{1}{2}\frac{(\rho/2)^l}{l!}\int_{-1}^{1} e^{i\rho t}(1-t^2)^l dt. \qquad (13)$$

As we did for the Gamma function, we rewrite it first as

$$j_l(\rho) = \frac{1}{2}\frac{(\rho/2)^l}{l!}\int_{-1}^{1} e^{i\rho t + l\ln(1-t^2)} dt. \tag{14}$$

The exponent

$$f(t) = i\rho t + l\ln(1-t^2) \tag{15}$$

is stationary at

$$f'(t) = i\rho + l\frac{-2t}{1-t^2} = 0, \tag{16}$$

or

$$t = i\frac{1}{\rho}\left(l \pm \sqrt{l^2-\rho^2}\right). \tag{17}$$

Here we assumed that $l > \rho$. Both extrema are on the positive imaginary axis. We are interested in the behavior when $l > \rho \gg 1$.

Let us now study the second derivative

$$f''(t) = l\frac{-2t(1-t^2)-4t^2}{(1-t^2)^2} = \begin{cases} -\frac{\rho^2\sqrt{l^2-\rho^2}}{(l-\sqrt{l^2-\rho^2})l} & t = i\frac{1}{\rho}\left(l-\sqrt{l^2-\rho^2}\right) \\ \frac{\rho^2\sqrt{l^2-\rho^2}}{(l+\sqrt{l^2-\rho^2})l} & t = i\frac{1}{\rho}\left(l+\sqrt{l^2-\rho^2}\right) \end{cases} \tag{18}$$

The extremum at $t = i\frac{1}{\rho}\left(l-\sqrt{l^2-\rho^2}\right)$ is close to the real axis and the direction of the "steepest descent" from the extremum is parallel to the real axis. Therefore we can deform the integration contour from -1 to 1 a little bit upwards on the complex plane to pass through this extremum parallel to the real axis, and pick up the Gaussian integral. However, the other extremum $t = i\frac{1}{\rho}\left(l+\sqrt{l^2-\rho^2}\right)$ is higher above the other extremum, and the direction of the "steepest descent" is along the *imaginary* axis. We cannot deform the contour smoothly to pick up the Gaussian integral along the imaginary axis. Therefore, we pick only the first extremum but not the second one. Then we can do Gaussian integral by extending the integration region to infinite and find

$$j_l(\rho) \simeq \frac{1}{2}\frac{1}{l!}e^{-l+\sqrt{l^2-\rho^2}}\left(\frac{l(l-\sqrt{l^2-\rho^2})}{\rho}\right)^l\left(\frac{2\pi l(l-\sqrt{l^2-\rho^2})}{\rho^2\sqrt{l^2-\rho^2}}\right)^{1/2} \tag{19}$$

If the Stirling's formula is used further for $l!$, it simplifies to

$$j_l(\rho) \simeq \frac{1}{2}e^{\sqrt{l^2-\rho^2}}\left(\frac{l-\sqrt{l^2-\rho^2}}{\rho}\right)^l\left(\frac{l-\sqrt{l^2-\rho^2}}{\rho^2\sqrt{l^2-\rho^2}}\right)^{1/2}$$

$$= \frac{1}{2} e^{\sqrt{l^2-\rho^2}} \left(\frac{\rho}{l+\sqrt{l^2+\rho^2}} \right)^l \left(\frac{1}{\sqrt{l^2-\rho^2}(l+\sqrt{l^2-\rho^2})} \right)^{1/2} \quad (20)$$

To obtain a similar asymptotic behavior for n_l, we need to go to the integral representation

$$h_l^{(\pm)}(\rho) = -i \frac{(\rho/2)^l}{l!} \int_{\pm 1}^{i\infty} e^{i\rho t} (1-t^2)^l dt. \quad (21)$$

We rewrite it as

$$h_l^{(\pm)}(\rho) = -i \frac{(\rho/2)^l}{l!} \int_{\pm 1}^{i\infty} e^{i\rho t + l \ln(1-t^2)} dt. \quad (22)$$

The extrema and the second derivatives are the same as before, Eqs. (17) and (18), respectively. The only difference now is that we can pick the extremum with plus sign along the imaginary axis, while we can pick only a *half* of the extremum with the negative sign, which gives the asymptotic behavior for j_l. In other words, the extremum with the plus sign is what gives n_l. Therefore

$$n_l(\rho) \simeq -i \frac{(\rho/2)^l}{l!} \int e^{i\rho t + l \ln(1-t^2)} dt, \quad (23)$$

where the integral is done along the imaginary axis around the extremum $t = i\frac{1}{\rho}\left(l + \sqrt{l^2-\rho^2}\right)$. The result is then

$$n_l(\rho) \simeq -i \frac{1}{l!} e^{-(l+\sqrt{l^2-\rho^2}} \left(\frac{l(l+\sqrt{l^2-\rho^2})}{\rho} \right)^l \left(\frac{2\pi l(l+\sqrt{l^2-\rho^2})}{\rho^2\sqrt{l^2-\rho^2}} \right)^{1/2} i, \quad (24)$$

and again using Stirling's formula for $l!$,

$$n_l(\rho) \simeq e^{-\sqrt{l^2-\rho^2}} \left(\frac{l+\sqrt{l^2-\rho^2}}{\rho} \right)^l \left(\frac{l+\sqrt{l^2-\rho^2}}{\rho^2\sqrt{l^2-\rho^2}} \right)^{1/2}. \quad (25)$$

When $l \gg \rho$, expressions Eq. (20,25) can be further simplified to

$$j_l(\rho) \simeq \frac{1}{2\sqrt{2}l} \left(\frac{e\rho}{2l} \right)^l, \quad (26)$$

$$n_l(\rho) \simeq \frac{\sqrt{2}}{\rho} \left(\frac{2l}{e\rho} \right)^l. \quad (27)$$

Clearly $n_l(\rho) \gg j_l(\rho)$.

3 Airy Function

Airy function is a solution to the equation

$$\left(\frac{d^2}{dx^2} - x\right) \mathrm{Ai}(x) = 0, \tag{28}$$

used around the classical turning points in the WKB approximation. An integral representation of the solution is

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t^3/3+xt)} dt. \tag{29}$$

Note that the integration contour can be deformed from the real axis to a path starting somewhere in the 3rd sextant $2\pi/3 < \arg t < \pi$ to the 1st sextant $0 < \arg t < \pi/3$.

The behavior for large x can be obtained by the steepest descent method. The exponent

$$f(t) = i\left(\frac{t^3}{3} + xt\right) \tag{30}$$

is extremum at

$$f'(t) = i(t^2 + x) = 0 \tag{31}$$

or

$$t = \pm\sqrt{-x}. \tag{32}$$

When $x > 0$, only the extremum at $t = i\sqrt{x}$ is naturally on the contour. Indeed, $f''(t) = 2it = -2\sqrt{x}$ has the steepest descent along the contour parallel to the real axis, while the extremum at $t = -i\sqrt{x}$ has the steepest descent along the imaginary axis. Therefore, we pick only the extremum $t = i\sqrt{x}$ and find

$$\mathrm{Ai}(x) \simeq \frac{1}{2\pi} \int e^{-\frac{2}{3}x^{3/2} - \sqrt{x}(t-i\sqrt{x})^2} = \frac{1}{2}\left(\frac{1}{\pi\sqrt{x}}\right)^{1/2} e^{-\frac{2}{3}x^{3/2}}. \tag{33}$$

On the other hand, when $x < 0$, the extrema are at $t = \pm\sqrt{-x}$, and both of them are naturally on the contour. At $t = -\sqrt{-x}$, the exponent is expanded as

$$f(t) = -i\frac{2}{3}x\sqrt{-x} - i\sqrt{-x}(t + \sqrt{-x})^2, \tag{34}$$

and hence the steepest descent is tilted by 45 degrees. Changing the variable by

$$t + \sqrt{-x} = e^{-i\pi/4} y, \tag{35}$$

the contribution of this extremum to the Airy function is

$$\frac{1}{2\pi} \int e^{-i\frac{2}{3}x\sqrt{-x} - \sqrt{-x}y^2} e^{-i\pi/4} dy = \frac{1}{2} e^{-i\pi/4} \left(\frac{\pi}{\sqrt{-x}} \right)^{1/2} e^{-i\frac{2}{3}x\sqrt{-x}}. \tag{36}$$

Similarly, the contribution from the extremum $t = \sqrt{-x}$ can be obtained using the steepest descent direction

$$t - \sqrt{-x} = e^{i\pi/4} y, \tag{37}$$

and we find

$$\frac{1}{2\pi} \int e^{i\frac{2}{3}x\sqrt{-x} - \sqrt{-x}y^2} e^{i\pi/4} dy = \frac{1}{2} e^{i\pi/4} \left(\frac{\pi}{\sqrt{-x}} \right)^{1/2} e^{i\frac{2}{3}x\sqrt{-x}}. \tag{38}$$

Adding Eqs. (36) and (38), we find

$$\mathrm{Ai}(x) \simeq \left(\frac{1}{\pi\sqrt{-x}} \right)^{1/2} \cos\left(\frac{2}{3} x\sqrt{-x} + \frac{\pi}{4} \right). \tag{39}$$

The differential equation Eq. (28) has another linearly independent solution. It is given in terms of the same integral along a different contour. One choice for the contour is C_1 from the 5th sextant $4\pi/3 < \arg t < 5\pi/3$ to the 1st sextant, or similarly C_2 from the 5th sextant to the 3rd sextant. We take the combination

$$\mathrm{Bi}(x) = \frac{-i}{2\pi} \left[\int_{C_1} e^{i(t^3/3 + xt)} dt + \int_{C_2} e^{i(t^3/3 + xt)} dt \right] \tag{40}$$

as the linearly independent solution. For $x < 0$ case, the extrema picked up on these contours are the same as Ai(x), but their relative sign is the opposite. On the other hand, for $x > 0$ case, we pick the different extremum $t = -i\sqrt{x}$ along the imaginary axis. In the end we find

$$\begin{aligned} \mathrm{Bi}(x) &\simeq \left(\frac{1}{\pi\sqrt{x}} \right)^{1/2} e^{x^{3/2}} \qquad (x \gg 0) \\ &\simeq \left(\frac{1}{\pi\sqrt{-x}} \right)^{1/2} \sin\left(\frac{2}{3} x\sqrt{-x} + \frac{\pi}{4} \right) \qquad (x \ll 0). \end{aligned} \tag{41}$$