Midterm

1. Particle on a circle

(a) Into the Lagrangian of a point particle $L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$, we substitute in $x = R\cos\theta$, $y = R\sin\theta$. Because the particle is always at the radius R, $\dot{x} = -R\,\dot{\theta}\sin\theta$, $\dot{y} = R\,\dot{\theta}\cos\theta$, and hence $L = \frac{1}{2} m R^2 \dot{\theta}^2$. The canonical momentum is given by its definition, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}$. The Hamiltonian is

$$H = p_\theta \dot{\theta} - L = p_\theta \frac{p_\theta}{m R^2} - \frac{1}{2} m R^2 \left(\frac{p_\theta}{m R^2}\right)^2$$
$$H = \frac{p_\theta{}^2}{2 m R^2}.$$

(b) The Heisenberg equation of motion is

$$i\hbar \frac{d}{dt}\theta = [\theta, H] = \left[\theta, \frac{p_\theta{}^2}{2 m R^2}\right] = i\hbar \frac{p_\theta}{m R^2} \implies \frac{d}{dt}\theta = \frac{p_\theta}{m R^2}$$
$$i\hbar \frac{d}{dt} p_\theta = [p_\theta, H] = \left[p_\theta, \frac{p_\theta{}^2}{2 m R^2}\right] = 0 \implies \frac{d}{dt} p_\theta = 0.$$

The solution to the second equation is simply that $p_\theta(t) = p_\theta(0)$ is conserved, and hence

$$\theta(t) = \theta(0) + \frac{p_\theta}{m R^2} t.$$

(c) The position−space wave function is $\langle\theta \mid p_\theta \mid k\rangle = \frac{\hbar}{i} \frac{\partial}{\partial\theta} \langle\theta \mid k\rangle = \hbar k \langle\theta \mid k\rangle$, and hence $\psi(\theta) = \langle\theta \mid k\rangle = N e^{i k \theta}$. To normalize it, we require $\int_0^{2\pi} |\psi(\theta)|^2 \, d\theta = 2\pi N^2 = 1$, and hence $N = 1\big/\sqrt{2\pi}$, $\psi(\theta) = \frac{1}{\sqrt{2\pi}} e^{i k \theta}$. In order to satisfy $\psi(\theta + 2\pi) = \psi(\theta)$, namely $e^{2\pi i k} = 1$, we need k to be an integer.

(d) The orthonormality is simply

$$\langle n \mid m\rangle = \int_0^{2\pi} \langle n \mid \theta\rangle \, d\theta \, \langle\theta \mid m\rangle = \int_0^{2\pi} \frac{e^{-i n \theta}}{\sqrt{2\pi}} \frac{e^{i m \theta}}{\sqrt{2\pi}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta.$$

When $n = m$, the integrand is unity, and hence $\langle n \mid n\rangle = \frac{1}{2\pi} 2\pi = 1$ and is normalized.

When $n \neq m$, $\langle n \mid m\rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta = \frac{1}{2\pi}\left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{1}{i(m-n)} - \frac{1}{i(m-n)}\right] = 0$ and are orthogonal.

(e) With the vector potential, there is an additional term to the Lagrangian

$$L_{\text{int}} = q\vec{A}\cdot\vec{v} = q\left(A_x \dot{x} + A_y \dot{y}\right) = q \frac{B}{2} \frac{d^2}{R^2} \left(y R \dot{\theta} \sin\theta + x R \dot{\theta} \cos\theta\right) = \frac{1}{2} q B d^2 \dot{\theta}.$$

Therefore, the total Lagrangian is

$$L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} q B d^2 \dot{\theta},$$

and hence the canonical momentum is modified as

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m R^2 \dot\theta + \frac{1}{2} q B d^2 .$$

The Hamitonian is therefore

$$H = p_\theta \dot\theta - L = p_\theta \frac{p_\theta - \frac{1}{2} q B d^2}{m R^2} - \frac{1}{2} m R^2 \left(\frac{p_\theta - \frac{1}{2} q B d^2}{m R^2}\right)^2 + \frac{1}{2} q B R^2 \frac{p_\theta - \frac{1}{2} q B d^2}{m R^2}$$
$$= \frac{1}{m R^2} \left(p_\theta{}^2 - \frac{1}{2} q B d^2 p_\theta - \frac{1}{2} p_\theta{}^2 + \frac{1}{2} q B d^2 p_\theta - \frac{1}{8} (q B d^2)^2 - \frac{1}{2} q B d^2 p_\theta + \frac{1}{4} (q B d^2)^2\right)$$
$$= \frac{1}{m R^2} \left(\frac{1}{2} p_\theta{}^2 - \frac{1}{2} q B d^2 p_\theta + \frac{1}{8} (q B d^2)^2\right.$$
$$H = \frac{1}{2 m R^2} \left(p_\theta - \frac{1}{2} q B d^2\right)^2 .$$

The eigenvalues of the canonical momentum is still $p_\theta = n \hbar$ because of the periodicity requirement, and hence the energy eigenvalues are

$$E_n = \frac{1}{2 m R^2} \left(n \hbar - \frac{1}{2} q B d^2\right)^2 .$$

Even though the particle never "sees" the magnetic field, the energy eigenvalues are affected by the vector potential, another manifestation of the Aharonov−Bohm effect. Note also that the result depends only on the total magnetic flux $\frac{q \Phi}{2 \hbar} = \frac{q B \pi d^2}{2 \pi \hbar}$ modulo integers.

2. Landau levels

(a) We expand the Hamiltonian in symmetric gauge,

$$H = \frac{1}{2 m} \left(\Pi_x{}^2 + \Pi_y{}^2\right) = \frac{1}{2 m} \left((p_x - \frac{e}{c} A_x)^2 + (p_y - \frac{e}{c} A_y)^2\right) = \frac{1}{2 m} \left(\left(p_x + \frac{e}{c} \frac{B}{2} y\right)^2 + \left(p_y - \frac{e}{c} \frac{B}{2} x\right)^2\right)$$
$$H = \frac{1}{2 m} (p_x{}^2 + p_y{}^2) + \frac{e^2 B^2}{8 m c^2} (x^2 + y^2) + \frac{e B}{2 m c} (y p_x - x p_y) = \frac{1}{2 m} (p_x{}^2 + p_y{}^2) + \frac{m \omega^2}{2} (x^2 + y^2) + \omega(y p_x - x p_y)$$

where $\omega = e B / 2 m c$. Using the standard oscillators

$$a_x = \sqrt{\frac{m \omega}{2 \hbar}} \left(x + \frac{i}{m \omega} p_x\right), \; a_y = \sqrt{\frac{m \omega}{2 \hbar}} \left(y + \frac{i}{m \omega} p_y\right),$$

we can recast our coordinates in oscillators,

$$x = \sqrt{\frac{\hbar}{2 m \omega}} (a_x{}^\dagger + a_x), \; p_x = i \sqrt{\frac{\hbar m \omega}{2}} (a_x{}^\dagger - a_x)$$
$$y = \sqrt{\frac{\hbar}{2 m \omega}} (a_y{}^\dagger + a_y), \; p_y = i \sqrt{\frac{\hbar m \omega}{2}} (a_y{}^\dagger - a_y) .$$

Substituting back into the Hamiltonian,

$$H = -\frac{1}{2 m} \frac{\hbar m \omega}{2} \left((a_x{}^\dagger)^2 - a_x{}^\dagger a_x - a_x a_x{}^\dagger + a_x{}^2 + (a_y{}^\dagger)^2 - a_y{}^\dagger a_y - a_y a_y{}^\dagger + a_y{}^2\right)$$
$$+ \frac{m \omega^2}{2} \frac{\hbar}{2 m \omega} \left((a_x{}^\dagger)^2 + a_x{}^\dagger a_x + a_x a_x{}^\dagger + a_x{}^2 + (a_y{}^\dagger)^2 + a_y{}^\dagger a_y + a_y a_y{}^\dagger + a_y{}^2\right)$$
$$+ \omega \left(i \frac{\hbar}{2}\right) (a_y{}^\dagger a_x{}^\dagger - a_y{}^\dagger a_x + a_y a_x{}^\dagger - a_y a_x - a_x{}^\dagger a_y{}^\dagger + a_x{}^\dagger a_y - a_x a_y{}^\dagger + a_x a_y)$$
$$= \frac{\hbar \omega}{2} (a_x{}^\dagger a_x + a_x a_x{}^\dagger + a_y{}^\dagger a_y + a_y a_y{}^\dagger + 2 i(a_x{}^\dagger a_y - a_y{}^\dagger a_x)) .$$

Using the commutation relation $[a, a^\dagger] = 1$,

$$H = \hbar\,\omega(a_x{}^\dagger\, a_x + a_y{}^\dagger\, a_y + 1 + i(a_x{}^\dagger\, a_y - a_y{}^\dagger\, a_x)).$$

(b) Similar to above, we can recast the Hamiltonian using $a_x = \frac{1}{\sqrt{2}}(a_z + a_{\bar{z}})$ and $a_y = \frac{1}{\sqrt{2}\,i}(a_z - a_{\bar{z}})$:

$$\begin{aligned}
H &= \hbar\omega\, \tfrac{1}{2}\Big((a_z{}^\dagger + a_{\bar{z}}{}^\dagger)(a_z + a_{\bar{z}}) + \left(\tfrac{1}{-i}\right)(a_z{}^\dagger - a_{\bar{z}}{}^\dagger)\left(\tfrac{1}{i}\right)(a_z - a_{\bar{z}}) + 2 \\
&\qquad + i(a_z{}^\dagger + a_{\bar{z}}{}^\dagger)\left(\tfrac{1}{i}\right)(a_z - a_{\bar{z}}) - i\left(\tfrac{1}{-i}\right)(a_z{}^\dagger - a_{\bar{z}}{}^\dagger)(a_z + a_{\bar{z}})\Big) \\
&= \hbar\omega\, \tfrac{1}{2}\,(2\,(a_z{}^\dagger\, a_z + a_{\bar{z}}{}^\dagger\, a_{\bar{z}}) + 2 \\
&\qquad + (a_z{}^\dagger\, a_z - a_z{}^\dagger\, a_{\bar{z}} + a_{\bar{z}}{}^\dagger\, a_z - a_{\bar{z}}{}^\dagger\, a_{\bar{z}}) + (a_z{}^\dagger\, a_z + a_z{}^\dagger\, a_{\bar{z}} - a_{\bar{z}}{}^\dagger\, a_z - a_{\bar{z}}{}^\dagger\, a_{\bar{z}})) \\
&= \hbar\omega\, \tfrac{1}{2}\,(2\,(a_z{}^\dagger\, a_z + a_{\bar{z}}{}^\dagger\, a_{\bar{z}}) + 2 + (2\,a_z{}^\dagger\, a_z - 2\,a_{\bar{z}}{}^\dagger\, a_{\bar{z}}))
\end{aligned}$$

$$H = \hbar\omega(2\,a_z{}^\dagger\, a_z + 1).$$

Now, applying this Hamiltonian to the presumed eigenstate,

$$H\,|\,n,\, m\rangle = \hbar\omega(2\,a_z{}^\dagger\, a_z + 1)\,\frac{(a_z{}^\dagger)^n\,(a_{\bar{z}}{}^\dagger)^m}{\sqrt{n!\,m!}}\,\Big|\,0,\,0\Big\rangle,$$

we need to commute the a_z through the daggered operators to the right. Let us evaluate the commutation relations:

$$\begin{aligned}
[a_z,\, a_z{}^\dagger] &= \tfrac{1}{2}[a_x + i\,a_y,\, a_x{}^\dagger - i\,a_y{}^\dagger] = \tfrac{1}{2}\,(1+1) = 1 \\
[a_z,\, a_{\bar{z}}{}^\dagger] &= \tfrac{1}{2}[a_x + i\,a_y,\, a_x{}^\dagger + i\,a_y{}^\dagger] = \tfrac{1}{2}\,(1-1) = 0\,.
\end{aligned}$$

Then, $\left[a_z,\, (a_z{}^\dagger)^n\right] = n(a_z{}^\dagger)^{n-1}$ and so

$$H\,|\,n,\, m\rangle = \hbar\omega(2\,n+1)\,|\,n,\, m\rangle$$

as desired.

(c) For an electron, the "Physical Constants" table from the PDG says the Bohr magneton is $\mu_B = \frac{e\,\hbar}{2\,m_e} = 5.79\times 10^{-11}$ MeV T^{-1}. The excitation energy is $\Delta E = \frac{e\,\hbar\,B}{m_e} = 1.16\times 10^{-3}$ eV for $B = 100$ kG $= 10\,T$. The corresponding thermal energy is $\Delta E / k = 1.16\times 10^{-3}$ eV$/8.62\times 10^{-5}$ eV $K^{-1} = 13.4\,K$. At temperatures below a few kelvin, practically all electrons populate the ground states.

(d) We have the requirement

$$\begin{aligned}
a_z\,|\,0,\,0\,\rangle &= \frac{1}{\sqrt{2}}\,(a_x + i\,a_y)\,\Big|\,0,\,0\Big\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{2\,\hbar}}\left(x + \frac{i}{m\,\omega}\,p_x\right) + i\,\sqrt{\frac{m\,\omega}{2\,\hbar}}\left(y + \frac{i}{m\,\omega}\,p_y\right)\right)\Big|\,0,\,0\Big\rangle \\
&= \sqrt{\frac{m\,\omega}{4\,\hbar}}\left(x + i\,y + \frac{1}{m\,\omega}\,(i\,p_x - p_y)\right)\Big|\,0,\,0\Big\rangle = 0
\end{aligned}$$

which in the position representation is

$$\sqrt{\frac{m\,\omega}{4\,\hbar}}\left(x + i\,y + \frac{1}{m\,\omega}\,(i\,(-i\,\hbar\,\partial_x) - (-i\,\hbar\,\partial_y))\right)\psi_{0,0} = \sqrt{\frac{m\,\omega}{4\,\hbar}}\left(x + i\,y + \frac{\hbar}{m\,\omega}\,(\partial_x + i\,\partial_y)\right)\psi_{0,0} = 0.$$

Similarly, for the requirement $a_{\bar{z}}\,|\,0,\,0\,\rangle = 0$, we have

$$\sqrt{\frac{m\,\omega}{4\,\hbar}}\left(x - i\,y + \frac{\hbar}{m\,\omega}\,(\partial_x - i\,\partial_y)\right)\psi_{0,0} = 0\;.$$

To make our lives easier, let us change coordinates from $(x,\, y)$ to $(z,\, \bar{z})$ where $z = x + i\,y$ and $\bar{z} = x - i\,y$. Then, $\partial = \frac{1}{2}\,(\partial_x - i\,\partial_y)$ and $\bar{\partial} = \frac{1}{2}\,(\partial_x + i\,\partial_y)$, such that $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial} z = 0$. So our equations become

$$\sqrt{\frac{m\,\omega}{4\,\hbar}}\left(z+\frac{2\,\hbar}{m\,\omega}\,\bar{\partial}\right)\psi_{0,0}=0,\ \sqrt{\frac{m\,\omega}{4\,\hbar}}\left(\bar{z}+\frac{2\,\hbar}{m\,\omega}\,\partial\right)\psi_{0,0}=0$$

which have the solution

$$\psi_{0,0}\propto e^{-m\,\omega\,z\,\bar{z}/2\,\hbar}=e^{-m\,\omega\,(x^2+y^2)/2\,\hbar}.$$

(e) For $n=0$ and $m\neq 0$, the first requirement is the same as above,

$$a_z\,\psi_{0,m}=\sqrt{\frac{m\,\omega}{4\,\hbar}}\left(z+\frac{2\,\hbar}{m\,\omega}\,\bar{\partial}\right)\psi_{0,m}=0\ ,$$

and this is clearly met by the form of the given wavefunction. For the second requirement, we use the number operator:

$$a_{\bar{z}}{}^{\dagger}\,a_{\bar{z}}\,|\,0,\,m\rangle=m\,|\,0,\,m\rangle\ .$$

In the position representation,

$$\begin{aligned}
a_{\bar{z}}{}^{\dagger}&=\frac{1}{\sqrt{2}}\,(a_x{}^{\dagger}+i\,a_y{}^{\dagger})=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{e\,B}{4\,\hbar\,c}}\left(x-\frac{2\,c}{e\,B}\,i\,p_x\right)+i\sqrt{\frac{e\,B}{4\,\hbar\,c}}\left(y-\frac{2\,c}{e\,B}\,i\,p_y\right)\right)\\
&=\sqrt{\frac{e\,B}{8\,\hbar\,c}}\left(x+i\,y-\frac{2\,c}{e\,B}\,(i\,p_x-p_y)\right)=\sqrt{\frac{e\,B}{8\,\hbar\,c}}\left(x+i\,y-\frac{2\,c}{e\,B}\,(i\,(-i\,\hbar\,\partial_x)+i\,\hbar\,\partial_y)\right)\\
&=\sqrt{\frac{e\,B}{8\,\hbar\,c}}\left(x+i\,y-\frac{2\,\hbar\,c}{e\,B}\,(\partial_x+i\,\partial_y)\right)=\sqrt{\frac{e\,B}{8\,\hbar\,c}}\left(z-\frac{4\,\hbar\,c}{e\,B}\,\bar{\partial}\right)
\end{aligned}$$

where ω has been substituted so as not to confuse the eigenvalue m with the mass m. Then, our requirement is

$$\begin{aligned}
&a_{\bar{z}}{}^{\dagger}\,a_{\bar{z}}\,|\,0,\,m\rangle=m\,|\,0,\,m\rangle\\
&\Longrightarrow\frac{e\,B}{8\,\hbar\,c}\left(z-\frac{4\,\hbar\,c}{e\,B}\,\bar{\partial}\right)\left(\bar{z}+\frac{4\,\hbar\,c}{e\,B}\,\partial\right)\psi_{0,m}=m\,\psi_{0,m}\\
&\Longrightarrow\left(\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}+\frac{1}{2}\,z\,\partial-\frac{1}{2}-\frac{1}{2}\,\bar{z}\,\bar{\partial}-2\left(\frac{\hbar\,c}{e\,B}\right)\bar{\partial}\,\partial\right)\psi_{0,m}=m\,\psi_{0,m}\ .
\end{aligned}$$

Applying the LHS operator to the given wavefunction, we find

$$\begin{aligned}
&\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}+\frac{1}{2}\,z\left(\frac{m}{z}-\frac{e\,B}{4\,\hbar\,c}\,\bar{z}\right)-\frac{1}{2}-\frac{1}{2}\,\bar{z}\left(-\frac{e\,B}{4\,\hbar\,c}\,z\right)-2\left(\frac{\hbar\,c}{e\,B}\right)\bar{\partial}\left(\frac{m}{z}-\frac{e\,B}{4\,\hbar\,c}\,\bar{z}\right)\\
&=\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}+\frac{m}{2}-\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}-\frac{1}{2}+\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}-2\left(\frac{\hbar\,c}{e\,B}\right)\left(-\frac{e\,B}{4\,\hbar\,c}+\left(\frac{m}{z}-\frac{e\,B}{4\,\hbar\,c}\,\bar{z}\right)\left(-\frac{e\,B}{4\,\hbar\,c}\,z\right)\right)\\
&=\frac{m}{2}-\frac{1}{2}+\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}+\frac{1}{2}+\frac{m}{2}-\frac{e\,B}{8\,\hbar\,c}\,z\,\bar{z}\\
&=m\,.
\end{aligned}$$

$\psi_{0,m}$ is indeed the wavefunction $\langle z,\,\bar{z}\,|\,0,\,m\rangle$.

Now let us find N:

$$\begin{aligned}
&\int|\,\psi_{0,m}\,|^2\,d\,x\,d\,y=N^2\int d\,x\,d\,y\,(x^2+y^2)^m\,e^{-e\,B\,(x^2+y^2)/2\,\hbar\,c}\\
&=N^2\int_0^\infty(2\,\pi\,r\,d\,r)\,r^{2\,m}\,e^{-e\,B\,r^2/2\,\hbar\,c}=1\\
&\Longrightarrow N=\left(\int_0^\infty(2\,\pi\,r\,d\,r)\,r^{2\,m}\,e^{-e\,B\,r^2/2\,\hbar\,c}\right)^{-1/2}\ .
\end{aligned}$$

Computing,

```
(Integrate[2 π r * r^(2 m) * Exp[- (e B)/(2 ℏ c) r^2],
   {r, 0, ∞}, Assumptions → {m > 0, e > 0, B > 0, ℏ > 0, c > 0}])^(-1/2)
```

$$\frac{1}{\sqrt{\pi}\sqrt{2^{1+m}\left(\frac{\hbar c}{B e}\right)^{1+m}\text{Gamma}[1+m]}}$$

We find $N = \left(\pi\, m!\left(\frac{2\hbar c}{e B}\right)^{m+1}\right)^{-1/2}$.

Another way to find the wave function is directly working out $\frac{(a_z{}^\dagger)^m}{\sqrt{m!}}\left|0, 0\right\rangle$. From the previous part,

$$\psi_{0,0} \propto e^{-m\omega(x^2+y^2)/2\hbar} = e^{-e B(x^2+y^2)/4\hbar c}.$$

The correct normalization is easily obtained by the Gaussian integral, and we find $\psi_{0,0} = \sqrt{\frac{e B}{2\pi\hbar c}}\, e^{-e B z \bar{z}/4\hbar c}$. The creation operator was worked out above,

$$a_z{}^\dagger = \sqrt{\frac{e B}{8\hbar c}}\left(z - \frac{4\hbar c}{e B}\bar{\partial}\right).$$

When we use this operator multiple times, there is never a derivative with respect to z, and hence $\bar{\partial}$ acts directly on $\psi_{0,0}$. Namely, $\bar{\partial} = -e B z/4\hbar c$ as long as it acts only on the ground state wave functions. Therefore,

$$a_z{}^\dagger = \sqrt{\frac{e B}{8\hbar c}}\left(z - \frac{4\hbar c}{e B}\frac{-e B z}{4\hbar c}\right) = \sqrt{\frac{e B}{2\hbar c}}\, z.$$

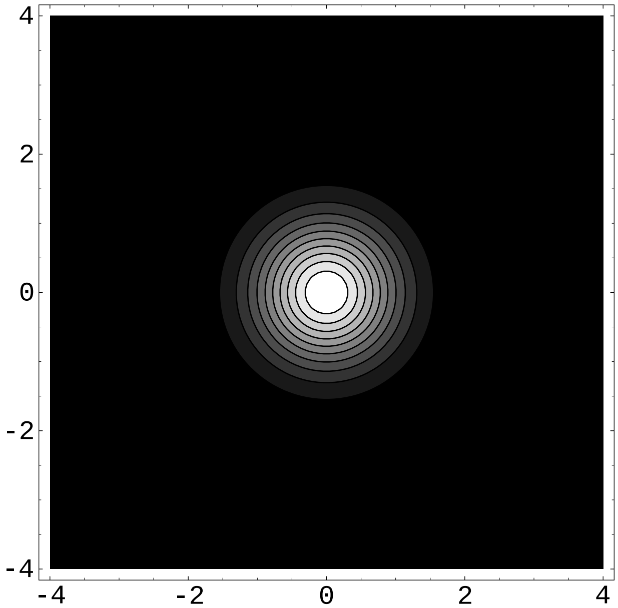
Using the definition,

$$\psi_{0,m} = \frac{(a_z{}^\dagger)^m}{\sqrt{m!}}\psi_{0,0} = \frac{1}{\sqrt{m!}}\left(\frac{e B}{2\hbar c}\right)^{m/2} z^m \sqrt{\frac{e B}{2\pi\hbar c}}\, e^{-e B z \bar{z}/4\hbar c}$$
$$= \left(\frac{1}{m!}\frac{1}{\pi}\left(\frac{e B}{2\hbar c}\right)^{m+1}\right)^{1/2} z^m\, e^{-e B z \bar{z}/4\hbar c}$$

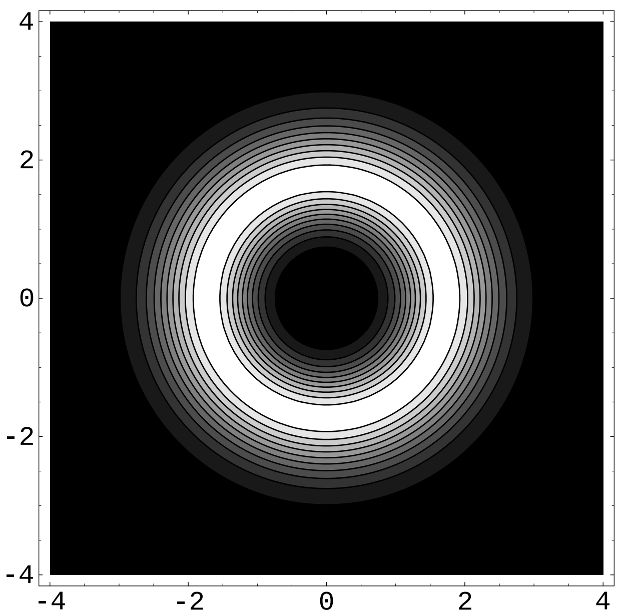
which is exactly the same result.

(f) Take $e B/2\hbar c = 1$. Then, $\psi_{0,m} = (m!\,\pi)^{-1/2} z^m\, e^{-\bar{z} z/2}$. Therefore,

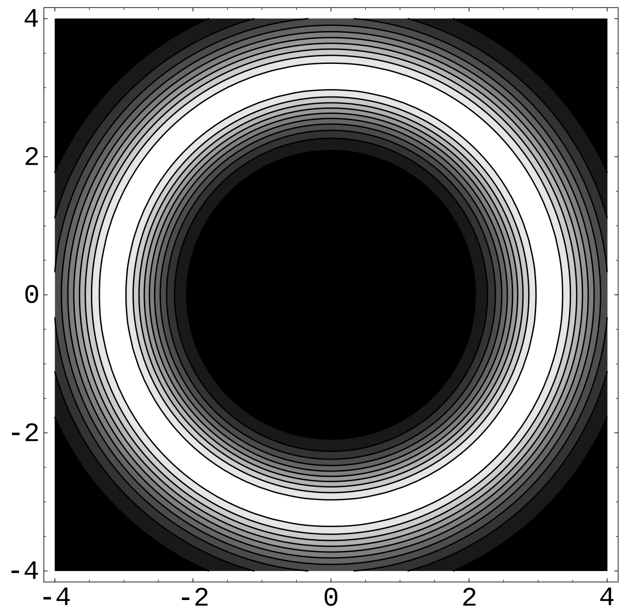
```
ContourPlot[(m! π)^-1 r^(2 m) E^(-r^2) /. {r -> √(x^2 + y^2)} /. {m → 0},
  {x, -4, 4}, {y, -4, 4}, PlotPoints → 100, PlotRange → {0, 1/π}];
```



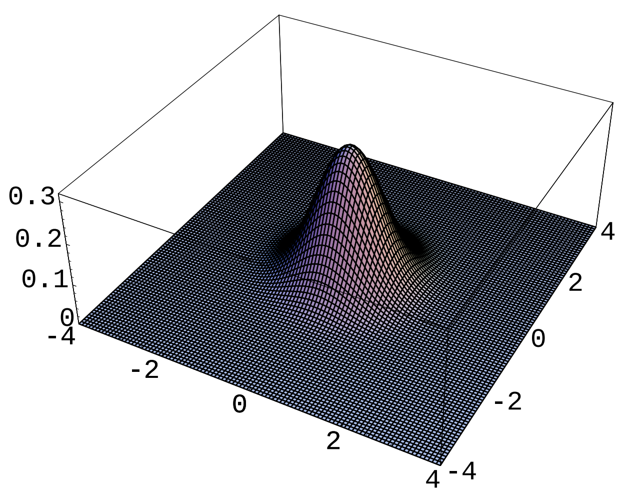

```
ContourPlot[(m! π)^-1 r^(2 m) E^(-r^2) /. {r -> √(x^2 + y^2)} /. {m → 3},
  {x, -4, 4}, {y, -4, 4}, PlotPoints → 100];
```



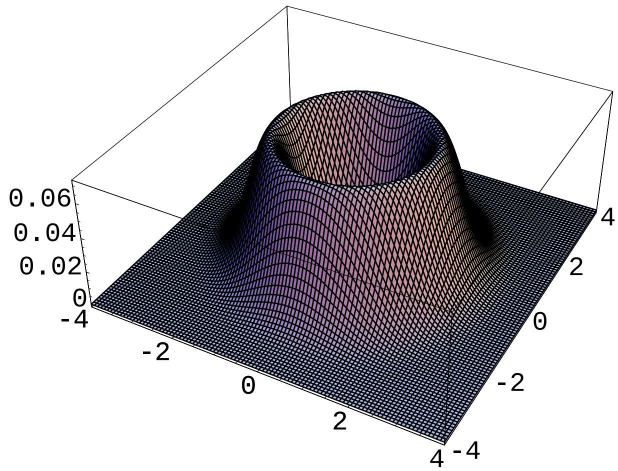
```
ContourPlot[(m! π)^-1 r^(2 m) E^(-r^2) /. {r -> √(x² + y²)} /. {m → 10},
  {x, -4, 4}, {y, -4, 4}, PlotPoints → 100];
```



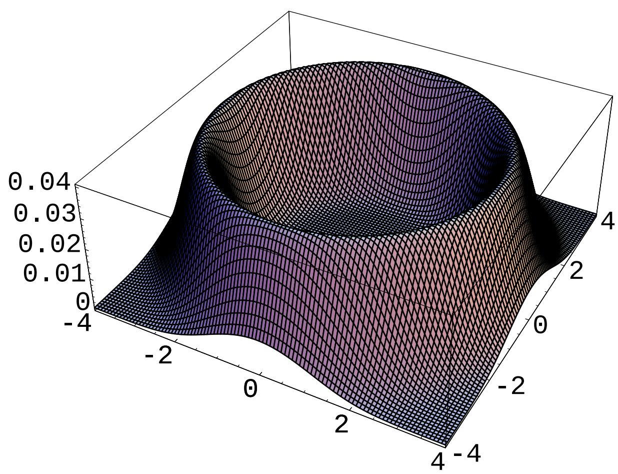

```
Plot3D[(m! π)^-1 r^(2 m) E^(-r^2) /. {r -> √(x² + y²)} /. {m → 0},
  {x, -4, 4}, {y, -4, 4}, PlotPoints → 100, PlotRange → {0, 1/π}];
```



```
Plot3D[(m ! π)^-1 r^(2 m) E^(-r^2) /. {r -> Sqrt[x^2 + y^2]} /. {m → 3},
  {x, -4, 4}, {y, -4, 4}, PlotPoints → 100];
```



```
Plot3D[(m ! π)^-1 r^(2 m) E^(-r^2) /. {r -> Sqrt[x^2 + y^2]} /. {m → 10},
  {x, -4, 4}, {y, -4, 4}, PlotPoints → 100];
```



(g) The wave function forms a ring further and further away from the orign for larger and larger n. If the system has a finite radius R, the ring goes outside the system for too large n. This sets a maximum value on n, and hence there are only a finite number of ground states. To obtain the maximum n, we require that the peak of the probability density is less than R.

```
Solve[D[(r^n E^(-e B r^2/(4 ℏ)))^2, r] == 0, r]
```

$$\left\{\left\{r \to -\frac{\sqrt{2}\,\sqrt{n}\,\sqrt{\hbar}}{\sqrt{B}\,\sqrt{e}}\right\}, \left\{r \to \frac{\sqrt{2}\,\sqrt{n}\,\sqrt{\hbar}}{\sqrt{B}\,\sqrt{e}}\right\}\right\}$$

For this radius to be inside the system, $\frac{2 n \hbar}{e B} < R^2$, and hence $n < \frac{e B R^2}{2 \hbar}$. The number of ground states is therefore $\frac{1}{2 \hbar} e B R^2$.

(h) Expanding the state in the Schrödinger picture,

$$|\,\psi_c\,;\,t\rangle = e^{-i H t/\hbar}\;|\,\psi_c\,\rangle = e^{-i H t/\hbar}\;e^{f\,a_z{}^{\dagger}}\;\left|0,\,0\right\rangle = \left(\sum \frac{e^{-i H t/\hbar}\,(f\,a_z{}^{\dagger})^l}{l!}\right)\left|0,\,0\right\rangle$$

$$= \left(\sum \frac{e^{-i\,\omega\,t\,(2\,l+1)}\,(f\,a_z{}^{\dagger})^l}{l!}\right)\left|0,\,0\right\rangle = e^{-i\,\omega\,t}\left(\sum (e^{-i\,2\,\omega\,t}\,)^l\;\frac{(f\,a_z{}^{\dagger})^l}{l!}\right)\left|0,\,0\right\rangle$$

$$= e^{-i\,\omega\,t}\;e^{f\,e^{-i\,2\,\omega\,t}\,a_z{}^{\dagger}}\;\left|0,\,0\right\rangle\;.$$

In the position representation

$$a_z{}^{\dagger} = \sqrt{\frac{m\,\omega}{4\,\hbar}}\left(\bar{z} - \frac{2\,\hbar}{m\,\omega}\,\partial\right) \equiv \frac{1}{2}\,k_0\left(\bar{z} - \frac{1}{k_0{}^2}\,\partial\right)$$

$$\langle z,\,\bar{z}\,|\,0,\,0\rangle = \frac{1}{\sqrt{\pi\left(\frac{\hbar}{m\,\omega}\right)}}\;e^{-m\,\omega\,z\,\bar{z}/2\,\hbar} \Rightarrow \sqrt{\frac{2\,k_0{}^2}{\pi}}\;e^{-k_0{}^2\,z\,\bar{z}}$$

so the wavefunction is

$$\langle z,\,\bar{z}\,|\,\psi_c\,;\,t\rangle \propto e^{-i\,\omega\,t}\;e^{f\,e^{-i\,2\,\omega\,t}\,k_0\,/2\,(\bar{z}-(1/k_0{}^2\,)\,\partial)}\;e^{-k_0{}^2\,z\,\bar{z}}$$

$$\Longrightarrow e^{-i\,\omega\,t}\;e^{k_0\;f\,e^{-i\,2\,\omega\,t}\;\bar{z}}\;e^{-k_0{}^2\;z\,\bar{z}}\;\;.$$

Making the coordinate transformation $\bar{z} \to \xi\,/\,f\,k_0$,

$$\langle \xi,\,\overline{\xi}\,|\,\psi_c\,;\,t\rangle \propto e^{-i\,\omega\,t}\;e^{\xi\,e^{-i\,2\,\omega\,t}}\;\;e^{-|\xi|^2\,/|f|^2}\;\;.$$

The position wavefunction is a 2D Gaussian profile whose center moves clockwise on a circular path, which is precisely cyclotron motion of frequency $2\,\omega = e\;\boldsymbol{B}\,/\,\boldsymbol{m}\,\boldsymbol{c}$ for an electron. (The first factor in the wavefunction is a time−dependent global phase due to zero−point energy.) To see this explicitly, we plot with $\omega = 1$, $f = 2$ and $\xi = x + i\,y$. (To see the output, run the command below; then, select the whole cell of plots, and create an animation by selecting the menu item "Cell−>Animate selected graphics" or by punching Control−y):

```
Table[ContourPlot[Exp[x Cos[2 t] - y Sin[2 t]] Exp[(-x^2 - y^2) / 4],
   {x, -4, 4}, {y, -4, 4}, PlotPoints → 64], {t, 0, 2 π, π / 10}];
```

3. Scalar Aharonov−Bohm

In this experiment, a magnetic field is applied for $\Delta t = 8\,\mu\,\text{sec}$ on neutrons whose magnetic moment is $\mu = -1.91\,\mu_N = -1.91\times 3.15\,10^{-14}\text{ MeV }T^{-1} = -6.02\times 10^{-14}\text{ MeV }T^{-1}$. The relative phase between two waves is (following Eq. (3)),

$$\Delta\Phi_{AB} = \frac{1}{\hbar}\,\mu\,B\,\Delta t = \frac{1}{6.58\times 10^{-22}\text{ MeV sec}}\left(-6.02\times 10^{-14}\text{ MeV }T^{-1}\right)B\times 8\times 10^{-6}\text{ sec} = 7.32\times 10^{2}\,(B/T) = 0.0732\,(B/\text{Gauss})$$

$$\Longrightarrow \Delta\Phi_{AB} = 7.32\times 10^{2}\,(B/T) = 0.0732\,(B/\text{Gauss}).$$

The fit to the data shown in Fig. 5 says $\Delta\Phi_{AB}/B = 0.0657/\text{Gauss}$, quite a good agreement with the expectation.

Fig. 2.4 in Sakurai's shows an electric potential, which has an electric field at the edges and hence forces. The reason why they chose to turn on and off the magnetic field is to avoid the possible criticism that a specially non−uniform field gives a non−uniform potential and hence a classical force. The purpose of the experiment, on the other hand, is to demonstrate the quantum phase in the absence of any classical force. Furthermore, they wanted to avoid the torque acting on the neutron spin, and therefore polarized the spins along the direction of the motion which is parallel to the magnetic field ("longitudinal polarization"). This way, they were sure that there is absolutely no classical force acting on neutrons, yet they showed the quantum phase, the scalar Aharonov−Bohm effect.

4. Neutrino flavor oscillation

(a) We construct the matrix

```
m2 = m02 {{1, 0}, {0, 1}} + dm2/2 {{-Cos[2 θ], Sin[2 θ]}, {Sin[2 θ], Cos[2 θ]}};
```

and solve for the eigenstates

```
Simplify[Eigensystem[m2], Assumptions → {dm2 > 0}]
```

```
{{-dm2/2 + m02, dm2/2 + m02}, {{-Cot[θ], 1}, {Tan[θ], 1}}}
```

We find the (normalized) eigenvalue/eigenvector pairs

$$\lambda_- = m_0{}^2 - \frac{\Delta m^2}{2},\quad v_- = \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$$

$$\lambda_+ = m_0{}^2 + \frac{\Delta m^2}{2},\quad v_+ = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

(b) The probability to measure state $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at time t is $|\langle\psi(0)\,|\,\psi(t)\rangle|^2$, where $\psi(t) = e^{-i H t/\hbar}\,\psi(0)$. If we write

$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\cos\theta\begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} + \sin\theta\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = -\cos\theta\; v_- + \sin\theta\, v_+,$$

then we may write

$$\psi(t) = -\cos\theta\; v_-\; e^{-i H_-\, t/\hbar} + \sin\theta\, v_+\; e^{-i H_+\, t/\hbar}$$

where $H_\pm = p\,c + \frac{c^3}{2\,p}\,\lambda_\pm = p\,c + \frac{c^3}{2\,p}\left(m_0{}^2 \pm \frac{\Delta\,m^2}{2}\right)$. Then, the probability is

$$|\langle\psi(0)\,|\,\psi(t)\rangle|^2 = |-\cos\theta\,((1,0)\cdot v_-)\,e^{-i\,H_-\,t/\hbar} + \sin\theta\,((1,0)\cdot v_+)\,e^{-i\,H_+\,t/\hbar}\,|^2$$
$$= \left|\cos^2\theta\,e^{-i\,H_-\,t/\hbar} + \sin^2\theta\,e^{-i\,H_+\,t/\hbar}\right|^2$$
$$= \left(\cos^2\theta\,e^{+i\,H_-\,t/\hbar} + \sin^2\theta\,e^{+i\,H_+\,t/\hbar}\right)\left(\cos^2\theta\,e^{-i\,H_-\,t/\hbar} + \sin^2\theta\,e^{-i\,H_+\,t/\hbar}\right)$$
$$= \cos^4\theta + \cos^2\theta\sin^2\theta(e^{i(H_+-H_-)\,t/\hbar} + e^{-i(H_+-H_-)\,t/\hbar}) + \sin^4\theta$$
$$= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos(H_+ - H_-)\,t/\hbar$$
$$= \left(\cos^2\theta + \sin^2\theta\right)^2 - 2\cos^2\theta\sin^2\theta + 2\cos^2\theta\sin^2\theta\cos(H_+ - H_-)\,t/\hbar$$
$$= 1 - 2\cos^2\theta\sin^2\theta(1 - \cos(H_+ - H_-)\,t/\hbar)$$
$$= 1 - \sin^2 2\,\theta\sin^2(H_+ - H_-)\,t/2\,\hbar$$

Inserting the eigenvalues and recognizing that neutrinos are ultra−relativistic,

$$|\langle\psi(0)\,|\,\psi(t)\rangle|^2 = 1 - \sin^2 2\,\theta\sin^2\frac{c^3}{4\,\hbar}\,\Delta\,m^2\,\frac{x}{E}\ .$$

This can be massaged further to use experimentally convenient units:

$$\langle\psi(0)\,|\,\psi(t)\rangle|^2 = 1 - \sin^2 2\,\theta\sin^2\frac{1}{4\,\hbar\,c}\,(\Delta\,m^2\,c^4)\,\frac{x}{E}$$
$$= 1 - \sin^2 2\,\theta\sin^2\frac{\text{GeV fm}}{4\,\hbar\,c}\,\frac{(\Delta\,m^2\,c^4)}{\text{eV}^2}\,\frac{x}{E}\,\frac{\text{GeV}}{\text{km}}$$

$$|\langle\psi(0)\,|\,\psi(t)\rangle|^2 = 1 - \sin^2 2\,\theta\sin^2 1.267\,\frac{(\Delta\,m^2\,c^4)}{\text{eV}^2}\,\frac{x}{E}\,\frac{\text{GeV}}{\text{km}}\ .$$

(c) Let us implement the expression to try to recreate the baseline and periodicity (in units of km/MeV) in the data:

```
prob = 1 - s22t Sin[1.267 dm2c4 (1000 s)]^2;

Plot[prob /. {s22t → .8, dm2c4 → 8 * 10^-5}, {s, 0, 80}];
```

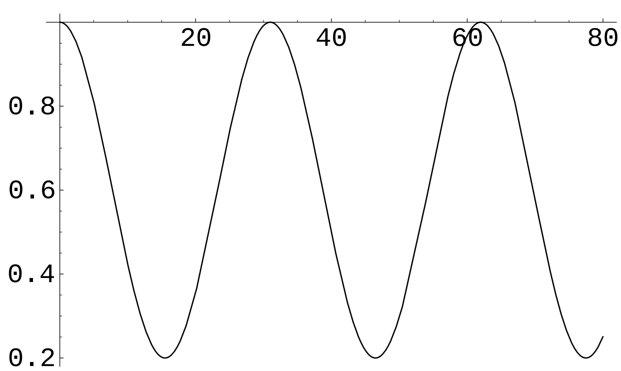


Fig. 3 in the paper shows a peak−to−peak wavelength of a bit over 30 km/MeV, which is reproduced nicely here with the value of $\Delta m^2 = 8\times10^{-5}\ \mathrm{eV}^2$ quoted in the paper. This is not surprising, as their quoted error is only $\pm 0.5\times10^{-5}\ \mathrm{eV}^2$.

The amplitude and center of oscillation are more problematic, as this is entirely dependent on θ. The authors quote a range of $0.33 < \tan^2\theta < 0.5$, which corresponds to $0.75 < \sin^2 2\theta < 0.9$; however, this is after including data from solar neutrinos, which puts strong constraints on θ as show in in Fig. 4(a). Looking at the 95% confidence range for just KamLAND, $0.1 < \tan^2\theta < 5$, or $0.33 < \sin^2 2\theta < 0.56$ passing through $\sin^2 = 1$. Examining Fig. 3, the peak and trough are at roughly 1.0 and 0.2 respectively, and this is reproduced above with the quoted center value of $\sin^2 2\theta = 0.8$ ($\tan^2\theta = 0.4$).