Final Exam (221A), due Dec 15, 4pm

1. The sodium D-lines are two separate emission lines for $3p \to 3s$ transitions. In a weak magnetic field, consider the Zeeman effect and discuss quantitatively

   (a) how the $3s$ and $3p$ levels are split due to the Zeeman effect, [10]

   (b) how the emission lines are split considering the selection rules. [10]

2. One useful way to use the Dyson series is to identify the energy shifts due to a perturbation, even when it is time-dependent.

   (a) When V is time-independent, work out $\langle i|U_I(t)|i\rangle$ to the second order, and identify $\Delta^{(1)}$, $\Delta^{(2)}$, and the wave function renormalization Z_i in the expansion of

   $$\begin{aligned}\langle i|U_I(t)|i\rangle &= Z_i e^{-i\Delta E t/\hbar} + \text{rapidly oscillating pieces} \\ &= Z_i \left[1 + \frac{-i}{\hbar}(\Delta_i^{(1)} + \Delta_i^{(2)})t + \frac{1}{2!}\left(\frac{-i}{\hbar}\Delta_i^{(1)} t\right)^2 + O(V^3)\right] \quad (1)\end{aligned}$$

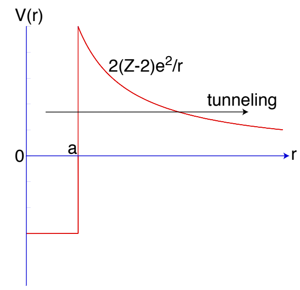
   and show that they agree with the results from the time-independent perturbation theory, Eqs. (5.1.42), (5.1.44), and (5.1.48b) in Sakurai. Note that this identification is done in the $t \to \infty$ limit where rapidly oscillating terms are dropped.[10]

   (b) Now consider a harmonic perturbation $V = -eE\cos\omega t$ where E is the size of the electric field. Work out the second-order energy shift.[10]

   (c) Use this formula to calculate the polarizability, and show that it is given by[10]

   $$\alpha(\omega) = -2e^2 \sum_{k\neq 0}^{\infty} \frac{|\langle k^{(0)}|z|1,0,0\rangle|^2 (E_0^{(0)} - E_k^{(0)})}{(E_0^{(0)} - E_k^{(0)})^2 - (\hbar\omega)^2}. \quad (2)$$

   (d) Discuss if the index of refraction increases or decreases for visible light ($\hbar\omega < |E_i - E_k|$) of shorter wave length $\lambda = 2\pi c/\omega$, and predict the order of colors in a rainbow.[10]



3. Nuclear α-decays $(A, Z) \to (A-2, Z-2) + \alpha$ have lifetimes ranging from nanoseconds (or shorter) to millions of years (or longer). This enormous range was understood by George Gamov by the exponential sensitivity to underlying parameters in tunneling phenomena. Consider $\alpha = {}^4\mathrm{He}$ as a point particle in the potential given schematically in the figure. The potential barrier is due to the Coulomb potential $2(Z-2)e^2/r$. The probability of tunneling is proportional to the so-called Gamov's transmission coefficients obtained in the WKB approximation

$$T = \exp\left[-\frac{2}{\hbar}\int_a^b \sqrt{2m(V(x)-E)}\,dx\right], \tag{3}$$

where a, b are the classical turning points. Work out numerically T for the following parameters: $Z = 92$ (Uranium), size of the nucleus $a = 5$ fm, and the kinetic energy of the α particle 1 MeV, 3 MeV, 10 MeV, 30 MeV. [15]

4. Discuss the impact of magnetic field on $2p$ states of the hydrogen atom due to two perturbation Hamiltonians

$$V_1 = \frac{1}{2m^2c^2}\frac{e^2}{r^3}\vec{L}\cdot\vec{S}, \qquad V_2 = -\frac{e}{2mc}(\vec{L} + g\vec{S})\cdot\vec{B} \tag{4}$$

Consider only states with $J_z = \frac{3}{2}\hbar$ and $\frac{1}{2}\hbar$.

   (a) When $V_1 \gg V_2$, *i.e.*, treat V_1 first, and then apply non-degenerate perturbation to V_2, both to its first order. [5]

   (b) When $V_2 \gg V_1$. [5]

   (c) When $V_1 \approx V_2$, *i.e.*, consider both of them at once with degenerate perturbation theory. Plot the energy levels as a function of the magnetic field in Tesla. Expand them in two limits and reproduce the results in (a) and (b). [15]