221A Lecture Notes
Convergence of Perturbation Theory

1 Asymptotic Series

An asymptotic series in a parameter ϵ of a function is given in a power series

$$f(\epsilon) = \sum_{n=0}^{\infty} f_n \epsilon^n \tag{1}$$

where the series actually *does not converge*. Instead, if you truncate the series at an order N

$$f_N(\epsilon) = \sum_{n=0}^{N} f_n \epsilon^n, \tag{2}$$

the difference between the true value $f(\epsilon)$ and and approximate expression $f_N(\epsilon)$ goes to zero $(f(\epsilon) - f_N(\epsilon))/\epsilon^N \to 0$ as $\epsilon \to 0$. In most examples, the perturbation theory gives an asymptotic series rather than a Taylor series. If it doesn't converge, why do we trust it?

In the lecture notes on the steepest descent method, an asymptotic expansion of the Gamma function $\Gamma(x)$ in inverse powers in x is discussed. The series is (obtained by `Series[Γ[x], {x,∞,10}]`),

$$\Gamma(x) = \sqrt{2\pi x}\, x^{x-1} e^{-x} \left(1 + \frac{1}{12\,x} + \frac{1}{288\,x^2} - \frac{139}{51840\,x^3} - \frac{571}{2488320\,x^4} \right. \tag{3}$$

$$+\frac{163879}{209018880\,x^5} + \frac{5246819}{75246796800\,x^6} - \frac{534703531}{902961561600\,x^7} - \frac{4483131259}{86684309913600\,x^8}$$

$$\left. +\frac{432261921612371}{514904800886784000\,x^9} + \frac{6232523202521089}{86504006548979712000\,x^{10}} + O(x^{-11}) \right). \tag{4}$$

It is an asymptotic series because its derivation ignores the exponentially suppressed tail when the integration region is changed from $x \in [0, \inf)$ to $(-\inf, \inf)$. Therefore, it misses corrections that behave as e^{-x} or e^{-x^2}. In fact, if you attempt to expand e^{-x} in power series of $1/x$, or equivalently, $g(\epsilon) = e^{-1/\epsilon}$ in power series of ϵ, you find that coefficient at each order vanishes.

$$g(0) = \lim_{\epsilon \to 0} e^{-1/\epsilon} = 0, \tag{5}$$

n	sum	n	sum	n	sum
0	0.922137	10	1.00047	20	−0.128483
1	0.922137	11	1.00053	21	−0.235955
2	0.998982	12	0.998768	22	12.1188
3	1.002180	13	0.998618	23	13.1524
4	0.999711	14	1.00452	24	−131.44
5	0.999499	15	1.00502	25	−143.527
6	1.000220	16	0.977792	26	1878.31
7	1.000290	17	0.975504	27	2047.24
8	0.999741	18	1.14106	28	−31242.9
9	0.999693	19	1.15495	29	−34023.3

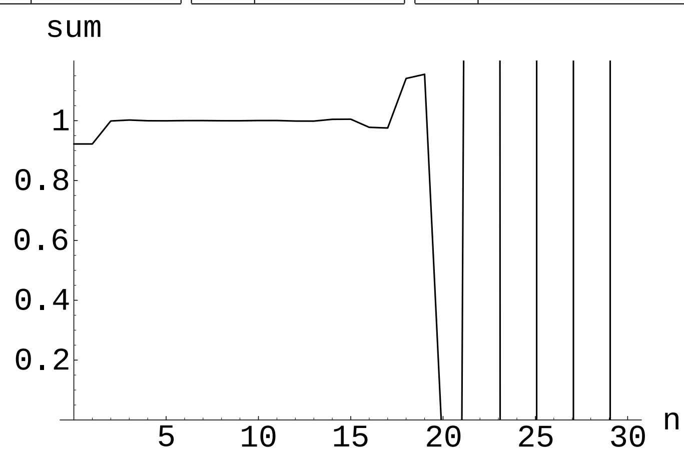

Table 1: The sum of the asymptotic series of $\Gamma(1)$ up to the n-th order.

$$g'(0) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} e^{-1/\epsilon} = 0, \tag{6}$$

$$g''(0) = \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon^4} - \frac{2}{\epsilon^3}\right) e^{-1/\epsilon} = 0, \tag{7}$$

and so on. At each finite order in Taylor expansion, the prefactor is always power-divergent while the exponential factor wins. Therefore the expansion of this function around $\epsilon = 0$ never "sees" the function at all. In other words, there is no information about this function at the origin. The exponentially suppressed correction is nominally "smaller" than the power-suppressed correction in power series expansion and ignored, while it comes back and haunts you when you evaluate it for the finite value of the expansion parameter.

On the other hand, this asymptotic series works quite well from the practical point of view. Even for $x = 1$, where an expansion in $1/x$ is not expected to be good at all, the series appears to converge to the true result $\Gamma(1) = 1$. The series approaches the true value up to the sixth order within the error of 0.00022, quite remarkable. But beyond the sixth order, the series starts to deviate from the true result, and eventually goes completely wild.

2 Convergence of the Perturbation Series

2.1 Infinite Radius of Convergence

Some perturbation series is convergent with an infinite radius of convergence. The simplest example is probably the harmonic oscillator with a linear term as the perturbation,

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 - x_0 x. \tag{8}$$

For simplicity, we take $m = \omega = \hbar = 1$. One can solve the system exactly, because

$$H = \frac{1}{2}p^2 + \frac{1}{2}(x - x_0)^2 - \frac{1}{2}x_0^2. \tag{9}$$

The ground state is only a shifted Gaussian, with the energy $E_0 = \frac{1}{2} - \frac{1}{2}x_0^2$. The perturbative expansion gives the second-order shift

$$\Delta^{(2)} = \frac{\langle 0^{(0)}| - x_0 x|1^{(0)}\rangle\langle 1^{(0)}| - x_0 x|0^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}} = \frac{x_0^2/2}{-1} = -\frac{1}{2}x_0^2, \tag{10}$$

reproducing the exact result. We used $x = (a + a^\dagger)/\sqrt{2}$ to obtain this result. It is not obvious, but all the higher order corrections vanish.

The ground-state wave function can be obtained by rewriting the Hamiltonian as

$$H = a^\dagger a + \frac{1}{2} - \frac{a + a^\dagger}{\sqrt{2}}x_0 = \left(a^\dagger - \frac{x_0}{\sqrt{2}}\right)\left(a - \frac{x_0}{\sqrt{2}}\right) - \frac{x_0^2}{2}. \tag{11}$$

The ground-state must satisfy

$$\left(a - \frac{x_0}{\sqrt{2}}\right)|0\rangle = 0, \tag{12}$$

and therefore it is a coherent state,

$$|0\rangle = e^{a^\dagger x_0/\sqrt{2}}|0^{(0)}\rangle e^{-x_0^2/4}. \tag{13}$$

In the perturbation theory, the correction to the ground-state wave function is given by Eq. (5.1.44) in Sakurai,

$$\begin{aligned} |0\rangle &= |0^{(0)}\rangle + |1^{(0)}\rangle \frac{\langle 1^{(0)}| - x_0 x |0^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}} \\ &\quad + |2^{(0)}\rangle \frac{\langle 2^{(0)}| - x_0 x |1^{(0)}\rangle \langle 1^{(0)}| - x_0 x |0^{(0)}\rangle}{(E_0^{(0)} - E_2^{(0)})(E_0^{(0)} - E_1^{(0)})} + O(x_0)^3 \\ &= |0^{(0)}\rangle + \frac{-x_0/\sqrt{2}}{-1}|1^{(0)}\rangle + \frac{x_0^2/\sqrt{2}}{2}|2^{(0)}\rangle + O(x_0)^3. \end{aligned} \tag{14}$$

This expression agrees with the Taylor series expansion of $e^{a^\dagger x_0/\sqrt{2}}|0^{(0)}\rangle$. However, this state is not properly normalized. It needs the wavefunction renormalization factor Eq. (5.1.48b) in Sakurai,

$$Z_0 = 1 - \frac{|\langle 1^{(0)}| - x_0 x |0^{(0)}\rangle|^2}{(-1)^2} = 1 - \frac{x_0^2}{2} \tag{15}$$

The normalization factor $e^{-x_0^2/4}$ agrees with $Z_0^{-1/2}$ up to this order.

Even if x_0 is large, the perturbation series converges.

2.2 Finite Radius of Convergence

Another simple example is again the harmonic oscillator with a quadratic term as the perturbation,

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{\epsilon}{2}x^2. \tag{16}$$

This is the example in Sakurai, pp. 294–296. The exact ground-state energy is

$$E_0 = \frac{1}{2}\sqrt{1+\epsilon} = \frac{1}{2}\left(1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)\right), \tag{17}$$

which agrees with the perturbative result. However, we know that the radius convergence of the Taylor series expansion of $\sqrt{1+\epsilon}$ is $|\epsilon| < 1$. In particular, there is a branch cut for $\epsilon < -1$ and it is clear that the series does not converge there.

There is a physical intuitive picture why the perturbation series does not converge for $|\epsilon| > 1$. For $\epsilon < -1$, the potential is actually upside down and there is no stable ground state. Therefore, the perturbation series *should not* converge!

2.3 Zero Radius of Convergence

The most famous example is the anharmonic oscillator,

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \lambda x^4. \tag{18}$$

The ground-state energy is expanded in the power series,

$$E_0 = \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \lambda^n A_n\right). \tag{19}$$

The coefficients A_n can be calculated using the recursion formula given by Carl M. Bender and Tai Tsun Wu, " Anharmonic Oscillator. II. A Study of Perturbation Theory in Large Order," *Phys. Rev.* **D 7**, 1620–1636 (1973), Section VI. They are given in Table 2.

In order to test the perturbative result, I also computed the ground-state energy numerically. I give examples for $\lambda = 0.01$ in Table 3 and 0.05 in Table 4. For the former, the series converges to the true result and appears stable up to the 40th order. For the latter, the series comes close to the true result at the sixth and seventh order, and then it starts to diverge.

We actually expect the perturbation series to have the zero radius of convergence. This is because a negative λ, no matter how small in the magnitude, leads to a potential unbounded from below and does not give a stable ground state. See Fig. 1. The "ground state" would tunnel through the potential barrier. The tunneling amplitude can be estimated using the WKB method if the barrier is sufficiently high, or equivalently, if $|\lambda|$ is sufficiently small. Neglecting the energy relative to the height of the potential barrier, the WKB amplitude of the tunneling is

$$\exp\left[-\int_0^{1/\sqrt{2|\lambda|}} \sqrt{2\left(\frac{1}{2}x^2 - |\lambda| x^4\right)}\, dx\right]. \tag{20}$$

Changing the variable $x = \xi/\sqrt{|\lambda|}$, it becomes

$$\exp\left[-\frac{1}{|\lambda|}\int_0^{1/\sqrt{2}} \sqrt{2\left(\frac{1}{2}\xi^2 - \xi^4\right)}\, d\xi\right] = e^{-1/6|\lambda|}. \tag{21}$$

Clearly, this effect cannot be obtained in the perturbation theory.

n	A_n	n	A_n
1	0.75000000	21	4.75124077×10^{28}
2	-2.62500000	22	-3.07579295×10^{30}
3	2.08125000×10^{1}	23	2.08301009×10^{32}
4	-2.41289062×10^{2}	24	-1.47290492×10^{34}
5	3.58098047×10^{3}	25	1.08552296×10^{36}
6	-6.39828135×10^{4}	26	-8.32483628×10^{37}
7	1.32973373×10^{6}	27	6.63329371×10^{39}
8	-3.14482147×10^{7}	28	-5.48392431×10^{41}
9	8.33541603×10^{8}	29	4.69782421×10^{43}
10	-2.44789407×10^{10}	30	-4.16502700×10^{45}
11	7.89333316×10^{11}	31	3.81734896×10^{47}
12	-2.77387770×10^{13}	32	-3.61299554×10^{49}
13	1.05564666×10^{15}	33	3.52778002×10^{51}
14	-4.32681068×10^{16}	34	-3.55023394×10^{53}
15	1.90081720×10^{18}	35	3.67917476×10^{55}
16	-8.91210175×10^{19}	36	-3.92301600×10^{57}
17	4.44255089×10^{21}	37	4.30055097×10^{59}
18	-2.34646431×10^{23}	38	-4.84327278×10^{61}
19	1.30915026×10^{25}	39	5.59961162×10^{63}
20	-7.69399985×10^{26}	40	-6.64186378×10^{65}

Table 2: The coefficients A_n in the perturbative expansion of the ground-state energy of the anharmonic oscillator.

Becker and Wu showed that the coefficients of the power series behaves as

$$A_n = \frac{(-1)^{n+1}\sqrt{6}}{\pi^{3/2}} 3^n \Gamma(n+\frac{1}{2}) \left(1 - \frac{95}{72}\frac{1}{n} + O(n^{-2})\right), \tag{22}$$

and hence grow faster than $n!$. No matter how small λ is, $A_n\lambda^n$ grows very fast for large n and makes the series non-convergent.

In general, it is typical that the perturbation theory has zero radius of convergence, and the perturbative series is only an asymptotic series. Nonetheless, as long as the expansion parameter is small, the series approaches the true result quite well at a finite order.

1	1.01500000	11	1.01451241	21	1.01451241	31	1.01451241
2	1.01447500	12	1.01451241	22	1.01451241	32	1.01451241
3	1.01451662	13	1.01451241	23	1.01451241	33	1.01451241
4	1.01451180	14	1.01451241	24	1.01451241	34	1.01451241
5	1.01451252	15	1.01451241	25	1.01451241	35	1.01451241
6	1.01451239	16	1.01451241	26	1.01451241	36	1.01451241
7	1.01451241	17	1.01451241	27	1.01451241	37	1.01451241
8	1.01451241	18	1.01451241	28	1.01451241	38	1.01451241
9	1.01451241	19	1.01451241	29	1.01451241	39	1.01451241
10	1.01451241	20	1.01451241	30	1.01451241	40	1.01451241

Table 3: The good apparent convergence of the series for $\lambda = 0.01$. The true result is 1.01451241 obtained by solving the Schrödinger equation numerically.

1	1.07500000	21	3.53914714×10^{1}
2	1.06187500	22	-1.11273814×10^{2}
3	1.06707813	23	3.85354663×10^{2}
4	1.06406201	24	-1.37048546×10^{3}
5	1.06630013	25	5.0997380×10^{3}
6	1.06430066	26	-1.97102171×10^{4}
7	1.06637837	27	7.91336015×10^{4}
8	1.06392148	28	-3.29450770×10^{5}
9	1.06717750	29	1.42062588×10^{6}
10	1.06239646	30	-6.33734492×10^{6}
11	1.07010479	31	2.92145041×10^{7}
12	1.05656047	32	-1.39028792×10^{8}
13	1.08233309	33	6.82346905×10^{8}
14	1.02951557	34	-3.45067145×10^{9}
15	1.14553227	35	1.79649554×10^{10}
16	0.87355644	36	-9.62098984×10^{10}
17	1.55143608	37	5.29602979×10^{11}
18	−0.23877459	38	-2.99434377×10^{12}
19	4.75523874	39	1.73769365×10^{13}
20	−9.91990573	40	-1.03437934×10^{14}

Table 4: The apparent convergence and then divergence of the series for $\lambda = 0.05$. The true result is 1.06528551 obtained by solving the Schrödinger equation numerically.

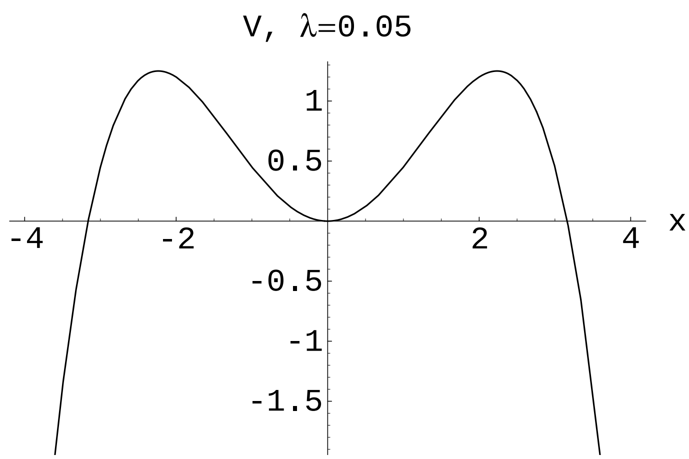

Figure 1: The potential of the anharmonic oscillator $V = \frac{1}{2}x^2 + \lambda x^4$ with $\lambda = 0.05$.