HW #6 (221A), due Oct 6, 4pm

1. Apply the WKB method to the harmonic oscillator.

   (a) Show that the energy levels comes out exactly.

   (b) Work out the wave functions for $n = 1$, $n = 10$, and $n = 20$, and compare to the exact results. (Hint: Mathematica knows Hermite polynomials `HermiteH[n,x]`.)

2. Bohr's correspondence principle states that in the limit of large quantum number the classical power radiated in the fundamental is equal to the product of quantum energy ($\hbar\omega_0$) and the reciprocal mean lifetime of the transition from principal quantum number n to $(n-1)$.

   (a) Show that the frequency of photon for a transition from principal quantum number $n \gg 1$ to $(n-k)$ ($k \ll n$) is the same as the frequency of the classical radiation from the circular orbit with the same energy. (Hint: the frequency of the classical radiation is given by the frequency of the revolution around the nucleus and its higher (interger multiple) modes.)

   (b) (optional) Using non-relativistic approximations, show that in a hydrogen-like atom the transition probability (reciprocal mean lifetime) for a transition from a circular orbit of principal quantum number n to $(n-1)$ is given classically by

   $$\frac{1}{\tau} = \frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4 \frac{mc^2}{\hbar}\frac{1}{n^5}. \tag{1}$$

   (c) For hydrogen compare the classical value above with the correct quantum-mechanical results for the mean lives of the transitions $2p \to 1s$ (1.6×10^{-9} sec), $4f \to 3d$ (7.3×10^{-8} sec), $6h \to 5g$ (6.1×10^{-7} sec).

3. The Hamitonian of a spin in the magnetic field is given by

   $$H = -g\frac{e}{2mc}\vec{s}\cdot\vec{B}. \tag{2}$$

   Assume $\vec{B} = (0, 0, B)$ is time-independent.

(a) Write down the Schrödinger equations for $|S_z;+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|S_z;-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and solve them to find the time dependence of these states.

(b) Write down the eigenstate $|S_x;+\rangle$ at $t = 0$ in S_z representation, and its time evolution.

(c) Calculate the time-dependence of the expectation values of S_x, S_y, and S_z in the above state to show that spin precesses.

4. (optional) The Maxwell equation (in Lorentz gauge) is

$$\left(\frac{n^2}{c^2}\frac{d^2}{dt^2} - \vec{\nabla}^2\right) A^0(\vec{x},t) = 0. \tag{3}$$

Consider it as a "Schrödinger equation" for the "light particle." Here, $n(\vec{x})$ is the index of refraction. Answer the following questions.

(a) Writing $A^0 = e^{iS/\hbar}$, use "WKB Approximation" to write down the "Hamilton–Jacobi" equation for $S(\vec{x},t)$.

(b) Show that it is the same as the Hamilton–Jacobi equation for a particle in a potential $V(\vec{x}) = -\frac{1}{2m}n(\vec{x})^2$ with zero energy up to an overall normalization factor.

(c) Assume that the index of refraction depends only on x. Then we can separate variables t and y (forget z in this problem). Solve for $\tilde{S}(x,p_y,E) = S(x,y,t) - p_y y + Et$ as a function of x in an integral expression in the way you normally do for Hamilton–Jacobi equation.

(d) Write down integral expressions for t and y.

(e) Specialize to the case where $n(x) = n_1$ for $x < 0$ and $n(x) = n_2$ for $x > 0$. Show that the trajectory of the "light particle" follows the usual rule of refraction.

Note The geometrical optics is none other than the "WKB approximation" for the Maxwell's equation.