HW #5

1. Harmonic Oscillator

(a)

We rewrite the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ using $a = \sqrt{\frac{m\omega}{2\hbar}} (x + i \frac{p}{m\omega})$, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} (x - i \frac{p}{m\omega})$. We first calculate

$$a^\dagger a = \frac{m\omega}{2\hbar} (x - i \frac{p}{m\omega}) (x + i \frac{p}{m\omega}) = \frac{m\omega}{2\hbar} \left(x^2 - \frac{i}{m\omega}[p, x] + \frac{p^2}{m^2 \omega^2}\right) = \frac{m\omega}{2\hbar} \left(x^2 - \frac{\hbar}{m\omega} + \frac{p^2}{m^2 \omega^2}\right).$$

Therefore,

$$\hbar\omega a^\dagger a = \frac{1}{2} m\omega^2 x^2 - \frac{1}{2}\hbar\omega + \frac{p^2}{2m},$$

and hence $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$.

(b)

The ground state condition $a|0\rangle = 0$ can be written in the position representation as

$$\langle x|a|0\rangle = \left\langle x \middle| \sqrt{\frac{m\omega}{2\hbar}} (x + i \frac{p}{m\omega}) \middle| 0\right\rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{1}{m\omega} \frac{\hbar}{i} \frac{d}{dx}\right) \langle x|0\rangle = 0,$$

and hence

$$\left(x + \frac{\hbar}{m\omega} \frac{d}{dx}\right)\psi_0(x) = 0.$$

This equation can be solved easily and we find

$$\psi_0(x) = N e^{-m\omega x^2/2\hbar}.$$

To normalize the wave function, we compute

$$\int_{-\infty}^{\infty} (e^{-m\omega x^2/2\hbar})^2 \, dx = \sqrt{\frac{\pi\hbar}{m\omega}}.$$

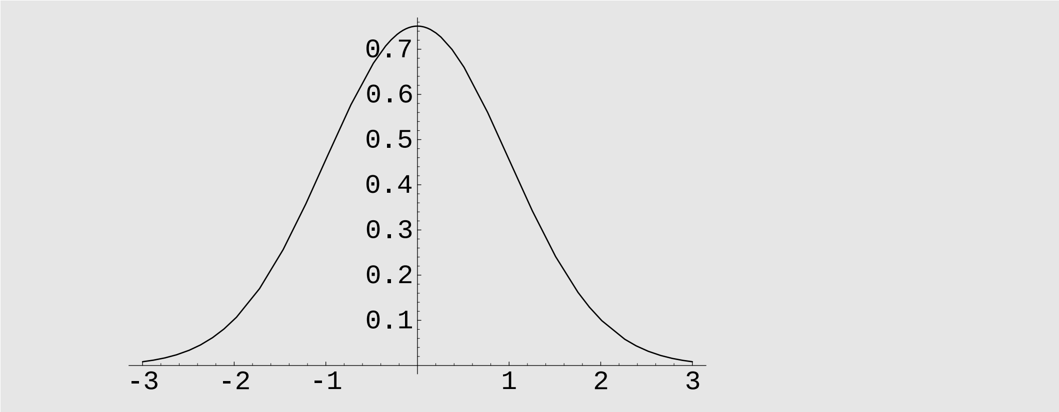
Therefore, the correctly normalized ground state wave function is

$$\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-m\omega x^2/2\hbar}.$$

The shape of the wave function is

```
ψ0[x_] := (m ω / (π ħ))^(1/4) E^(-m ω x^2 / 2 ħ)
```

```
Plot[ψ0[x] /. {m → 1, ω → 1, ℏ → 1}, {x, -3, 3}];
```



(c)

The first excited state is given by $|1\rangle = a^\dagger\,|0\rangle$, and its position representation by

$$\langle x\,|\,1\rangle = \langle x\,|\,a^\dagger\,|\,0\rangle = \sqrt{\frac{m\,\omega}{2\,\hbar}}\left(x - i\,\frac{1}{m\,\omega}\,\frac{\hbar}{i}\,\frac{d}{d\,x}\right)\langle x\,|\,0\rangle$$

$$= \sqrt{\frac{m\,\omega}{2\,\hbar}}\,(x+x)\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} e^{-m\,\omega\,x^2/(2\,\hbar)} = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\sqrt{\frac{m\,\omega}{2\,\hbar}}\,2\,x\,e^{-m\,\omega\,x^2/(2\,\hbar)}$$

Its shape is

$$\psi_1[x\_] := \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\sqrt{\frac{m\,\omega}{2\,\hbar}}\;2\,x\,E^{-m\,\omega\,x^2/(2\,\hbar)}$$

```
Plot[ψ1[x] /. {m → 1, ω → 1, ℏ → 1}, {x, -3, 3}];
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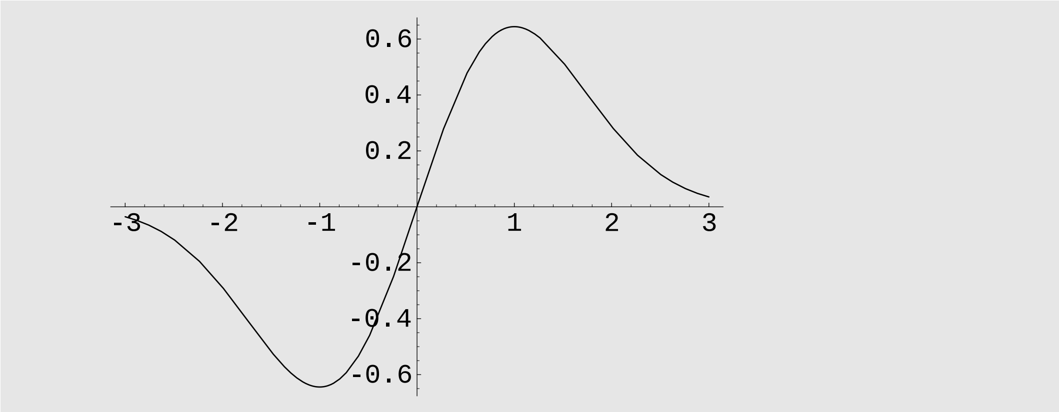

Check that it is properly normalized:

$$\text{Integrate}\left[\psi_1[x]^2,\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \text{Re}\left[\frac{m\,\omega}{\hbar}\right] > 0\right]$$

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1
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The second excited state is given by $\sqrt{2}\,|2\rangle = a^\dagger\,|1\rangle$, and its position representation by

$$\langle x|2\rangle = \frac{1}{\sqrt{2}}\langle x|a^\dagger|0\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{1}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\right)\langle x|1\rangle$$

Its shape is

$$\text{Simplify}\left[\frac{1}{\sqrt{2}}\sqrt{\frac{m\,\omega}{2\,\hbar}}\left(x\,\psi_1[x] - \frac{\hbar}{m\,\omega}\,\text{D}[\psi_1[x],\,x]\right)\right]$$

$$\frac{e^{-\frac{m x^2 \omega}{2\hbar}}\,(2\,m\,x^2\,\omega - \hbar)\,\left(\frac{m\,\omega}{\hbar}\right)^{1/4}}{\sqrt{2}\,\pi^{1/4}\,\hbar}$$

$$\psi_2[x\_] := \frac{e^{-\frac{m x^2 \omega}{2\hbar}}\,(2\,m\,x^2\,\omega - \hbar)\,\left(\frac{m\,\omega}{\hbar}\right)^{1/4}}{\sqrt{2}\,\pi^{1/4}\,\hbar}$$

```
Plot[ψ2[x] /. {m → 1, ω → 1, ħ → 1}, {x, -3, 3}];
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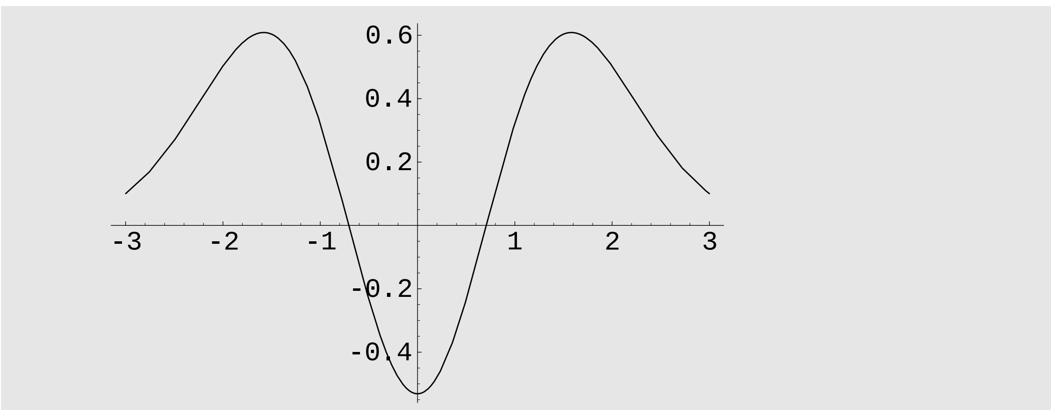

Check that it is properly normalized:

$$\text{Integrate}\left[\psi_2[x]^2,\,\{x,\,-\infty,\,\infty\},\,\text{Assumptions} \to \text{Re}\left[\frac{m\,\omega}{\hbar}\right] > 0\right]$$

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1
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(d)

From the definitions of the annihilation and creation operators, we can solve for x,

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger).$$

Starting with the expectation values,

$$\langle x\rangle = \langle n|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|a + a^\dagger|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right).$$

Because of the orthonormality of the Hamiltonian eigenstates $\langle n|m\rangle = \delta_{n,m}$,

$\langle x \rangle = 0.$

Moving on to the variance,

$$\langle x^2 \rangle = \frac{\hbar}{2 m \omega} \left\langle n \middle| (a + a^\dagger)^2 \middle| n \right\rangle = \frac{\hbar}{2 m \omega} \langle n | a^\dagger a + a a^\dagger | n \rangle,$$
$$= \frac{\hbar}{2 m \omega} \langle n | 2 N + [a, a^\dagger] | n \rangle = \frac{\hbar}{2 m \omega} (2 n + 1).$$

$\left\langle (\Delta x)^2 \right\rangle = \frac{\hbar}{2 m \omega} (2 n + 1).$

From the definitions of the annihilation and creation operators, we can solve for p,

$$p = -i \sqrt{\frac{\hbar m \omega}{2}} (a - a^\dagger).$$

Together with the expression for x from the previous problem, you can easily verify $[x, p] = i \hbar$. Starting with the expectation values,

$$\langle p \rangle = \langle n | p | n \rangle = -i \sqrt{\frac{\hbar m \omega}{2}} \langle n | a - a^\dagger | n \rangle,$$

$\langle p \rangle = 0 .$

Moving on to the variance,

$$\langle p^2 \rangle = -\frac{\hbar m \omega}{2} \left\langle n \middle| (a - a^\dagger)^2 \middle| n \right\rangle = \frac{\hbar m \omega}{2} \langle n | a^\dagger a + a a^\dagger | n \rangle,$$
$$= \frac{\hbar m \omega}{2} \langle n | 2 N + [a, a^\dagger] | n \rangle = \frac{\hbar m \omega}{2} (2 n + 1).$$

$\left\langle (\Delta p)^2 \right\rangle = \frac{\hbar m \omega}{2} (2 n + 1) .$

Therefore,

$$(\Delta x) (\Delta p) = \frac{\hbar}{2} (2 n + 1).$$

The ground state $n = 0$ is a minimum uncertainty state, while the excited states have larger uncertainties.

(e)

There are many ways to show this.

First of all, using the relation $a^\dagger\,|n\rangle = \sqrt{n+1}\,\left|n+1\right\rangle$, we can show that

$$(a^\dagger)^n\,\left|0\right\rangle = \sqrt{n!}\,\left|n\right\rangle$$

by recursion. It obviously holds for $n = 1$. If it holds for n, the $n+1$-th one is

$$(a^\dagger)^{n+1}\,\left|0\right\rangle = a^\dagger\,(a^\dagger)^n\,\left|0\right\rangle = a^\dagger\,\sqrt{n!}\,\left|n\right\rangle = \sqrt{n!}\,\sqrt{n+1}\,\left|n+1\right\rangle$$
$$= \sqrt{(n+1)!}\,\left|n+1\right\rangle$$

, and it holds again. Therefore, it holds for all n.

Using the definition

$$|\,f\rangle = e^{f\,a^\dagger}\,\left|0\right\rangle e^{-|f|^2/2} = \sum_{n=0}^{\infty} \frac{f^n}{n!}\,(a^\dagger)^n\,\left|0\right\rangle e^{-|f|^2/2},$$

we can write it as

$$|\,f\rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}}\,\left|n\right\rangle e^{-|f|^2/2}.$$

Then

$$a\,|\,f\rangle = a\sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}}\,\left|n\right\rangle e^{-|f|^2/2} = \sum_{n=1}^{\infty} \frac{f^n}{\sqrt{n!}}\,\sqrt{n}\,\left|n-1\right\rangle e^{-|f|^2/2},$$

where the summation is now taken only from $i = 1$ because $i = 0$ term vanishes by $a\,|\,0\rangle = 0$. Continuing on,

$$a\,|\,f\rangle = \sum_{n=1}^{\infty} \frac{f^n}{\sqrt{(n-1)!}}\,\left|n-1\right\rangle e^{-|f|^2/2} = \sum_{m=0}^{\infty} \frac{f^{m+1}}{\sqrt{m!}}\,\left|m\right\rangle e^{-|f|^2/2},$$

where the dummy variable was chgned to $m = n - 1$. Pulling one factor of f out of the sum,

$$a\,|\,f\rangle = f\sum_{m=0}^{\infty} \frac{f^m}{\sqrt{m!}}\,\left|m\right\rangle e^{-|f|^2/2} = f\,|\,f\rangle.$$

Another way to show the same result is by first showing the relation

$$\left[a,\,(a^\dagger)^n\right] = n(a^\dagger)^{n-1}.$$

It obviously holds for $n = 1$. If it holds for n, the $n+1$-th one is

$$\left[a,\,(a^\dagger)^{n+1}\right] = \left[a,\,a^\dagger\,(a^\dagger)^n\right] = [a,\,a^\dagger]\,(a^\dagger)^n + a^\dagger\left[a,\,(a^\dagger)^n\right]$$
$$= (a^\dagger)^n + a^\dagger\,n(a^\dagger)^n = (n+1)\,(a^\dagger)^{n+1}$$

and hence it holds as well. Therefore, it holds for any n.

Starting with the definition

$$|\,f\rangle = e^{f\,a^\dagger}\,\left|0\right\rangle e^{-|f|^2/2} = \sum_{n=0}^{\infty} \frac{f^n}{n!}\,(a^\dagger)^n\,\left|0\right\rangle e^{-|f|^2/2},$$
$$a\,|\,f\rangle = a\sum_{n=0}^{\infty} \frac{f^n}{n!}\,(a^\dagger)^n\,\left|0\right\rangle e^{-|f|^2/2} = \sum_{n=0}^{\infty} \frac{f^n}{n!}\left[a,\,(a^\dagger)^n\right]\left|0\right\rangle e^{-|f|^2/2},$$

Here, we used the fact $a\,|\,0\rangle = 0$. Using the relation shown above,

$$a\,|\,f\rangle = \sum_{n=1}^{\infty} \frac{f^n}{n!}\,n\,(a^\dagger)^{n-1}\,\left|0\right\rangle e^{-|f|^2/2} = \sum_{n=1}^{\infty} \frac{f^n}{(n-1)!}\,(a^\dagger)^{n-1}\,\left|0\right\rangle e^{-|f|^2/2},$$

where the summation is now taken only from $i = 1$ because $i = 0$ term vanishes by $[a,\,1] = 0$. Changing the dummy variable to $m = n - 1$,

$$a\,|\,f\rangle = \sum_{m=0}^{\infty} \frac{f^{m+1}}{m!}\,(a^\dagger)^m\,\left|0\right\rangle e^{-|f|^2/2},$$

$$a\,|\,f\rangle = f\sum_{m=0}^{\infty} \frac{f^m}{m!}\,(a^\dagger)^m\,\left|0\right\rangle e^{-|f|^2/2} = f\,|\,f\rangle.$$

Finally, we verify the normalization,

$$\langle f \mid f\rangle = e^{-|f|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(f^*)^m}{m!} \frac{f^n}{n!} \left\langle 0 \middle| a^m (a^\dagger)^n \middle| 0\right\rangle.$$

Consider a acting on the left and $a^\dagger$ acting on the right, again by orthonormality we must have $m = n$, so

$$\langle f \mid f\rangle = e^{-|f|^2} \sum_{n=0}^{\infty} \frac{|f|^2}{n!\,n!} \left\langle n \middle| \sqrt{n!}\,\sqrt{n!} \middle| n\right\rangle$$

$$= e^{-|f|^2} \sum_{n=0}^{\infty} \frac{|f|^2}{n!} \langle n \mid n\rangle = e^{-|f|^2}\; e^{|f|^2}$$

$$\langle f \mid f\rangle = 1.$$

(f)

Using the result from (d),

$$\langle f \mid x \mid f\rangle = \left\langle f \middle| \sqrt{\frac{\hbar}{2 m \omega}} (a + a^\dagger) \middle| f\right\rangle = \sqrt{\frac{\hbar}{2 m \omega}} (f + f^*) = \sqrt{\frac{\hbar}{2 m \omega}}\, 2\,\mathrm{Re}(f).$$

Here, we used the fact $\langle f \mid a^\dagger = \langle f \mid f^*$, obtained by taking the hermitian conjugate of $a \mid f\rangle = f \mid f\rangle$. Similarly, using the result from (e),

$$\langle f \mid p \mid f\rangle = \left\langle f \middle| -i \sqrt{\frac{\hbar m \omega}{2}} (a - a^\dagger) \middle| f\right\rangle = -i \sqrt{\frac{\hbar m \omega}{2}} (f - f^*) = \sqrt{\frac{\hbar m \omega}{2}}\, 2\,\mathrm{Im}(f).$$

Now on the variance,

$$\langle f \mid x^2 \mid f\rangle = \left\langle f \middle| \frac{\hbar}{2 m \omega} (a + a^\dagger)^2 \middle| f\right\rangle = \frac{\hbar}{2 m \omega} \left\langle f \middle| a^2 + a\,a^\dagger + a^\dagger\, a + (a^\dagger)^2 \middle| f\right\rangle$$

$$= \frac{\hbar}{2 m \omega} \left\langle f \middle| a^2 + [a,\ a^\dagger] + 2\,a^\dagger\, a + (a^\dagger)^2 \middle| f\right\rangle = \frac{\hbar}{2 m \omega} \left(f^2 + 1 + 2\, f^*\, f + (f^*)^2\right)$$

$$= \frac{\hbar}{2 m \omega} \left((f + f^*)^2 + 1\right)$$

and hence

$$\left\langle (\Delta x)^2 \right\rangle = \frac{\hbar}{2 m \omega}.$$

Similarly,

$$\langle f \mid p^2 \mid f\rangle = \left\langle f \middle| -\frac{\hbar m \omega}{2} (a - a^\dagger)^2 \middle| f\right\rangle = \frac{\hbar m \omega}{2} \left\langle f \middle| -a^2 + a\,a^\dagger + a^\dagger\, a - (a^\dagger)^2 \middle| f\right\rangle$$

$$= \frac{\hbar m \omega}{2} \left\langle f \middle| -a^2 + [a,\ a^\dagger] + 2\,a^\dagger\, a - (a^\dagger)^2 \middle| f\right\rangle = \frac{\hbar m \omega}{2} \left(-f^2 + 1 + 2\, f^*\, f - (f^*)^2\right)$$

$$= \frac{\hbar m \omega}{2} \left(-(f - f^*)^2 + 1\right)$$

and hence

$$\left\langle (\Delta p)^2 \right\rangle = \frac{\hbar m \omega}{2}.$$

Therefore,

$$(\Delta x)\,(\Delta p) = \frac{\hbar}{2}$$

and hence the coherent state is a minimum uncertainty state for any f.

(g) [optional]

We take the solution from Eq. 2

```
ψf[x_] = (m ω/(π ℏ))^(1/4) Exp[-(Sqrt[m ω/(2 ℏ)] x - f)^2 + 1/2 (f^2 - Abs[f]^2)];
```

and apply the operator a as in part (b)

```
resa = Sqrt[m ω/(2 ℏ)] (x ψf[x] + i 1/(m ω) ℏ/i ∂x ψf[x]);
```

. The desired result is

```
resd = f ψf[x];
```

. We see that they are indeed the same

```
Simplify[resa - resd]
```

```
0
```

(h)

The Heisenberg equation of motion is

$$i\hbar \frac{d}{dt} x = [x, H] = \left[x, \frac{p^2}{2m}\right] = i\hbar \frac{p}{m},$$

$$i\hbar \frac{d}{dt} p = [p, H] = \left[p, \frac{1}{2} m\omega^2 x^2\right] = -i\hbar m\omega^2 x.$$

There are many ways to solve these coupled equations. One way is to use the exponential of a matrix. Write the equations in the matrix form,

$$\frac{d}{dt}\begin{pmatrix} m\omega x \\ p \end{pmatrix} = \begin{pmatrix} \omega p \\ -m\omega^2 x \end{pmatrix} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} m\omega x \\ p \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} m\omega x \\ p \end{pmatrix}(t) = \exp\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} i\omega t\right)\begin{pmatrix} m\omega x \\ p \end{pmatrix}(0) = \exp(\sigma_2\, i\omega t)\begin{pmatrix} m\omega x \\ p \end{pmatrix}(0).$$

Here, σ_2 is one of the Pauli matrices. The exponential factor can be worked out using its Taylor expansion,

$$\exp(\sigma_2\, i\omega t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_2{}^n (i\omega t)^n .$$

It is easy to check that $\sigma_2{}^2 = 1$, and hence $\sigma_2{}^{\text{even}} = 1$, $\sigma_2{}^{\text{odd}} = \sigma_2$. Therefore,

$$\exp(\sigma_2\, i\omega t) = \sum_{n=0,\text{ even}}^{\infty} \frac{1}{n!} 1 (i\omega t)^n + \sum_{n=0,\text{ odd}}^{\infty} \frac{1}{n!} \sigma_2 (i\omega t)^n$$

$$= 1 \cos\omega t + i\sigma_2 \sin\omega t = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

We find the solution

$$\begin{pmatrix} m\omega x \\ p \end{pmatrix}(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} m\omega x \\ p \end{pmatrix}(0)$$

$$= \begin{pmatrix} m\omega x(0)\cos\omega t + p(0)\sin\omega t \\ -m\omega x(0)\sin\omega t + p(0)\cos\omega t \end{pmatrix}.$$

Using this solution, we calculate the expectation values,

$$\langle x\rangle(t) = \langle f \,|\, x(t) \,|\, f\rangle = \left\langle f \,\middle|\, x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t \,\middle|\, f\right\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\; 2\,\mathrm{Re}(f)\cos\omega t + \frac{1}{m\omega}\sqrt{\frac{\hbar m\omega}{2}}\; 2\,\mathrm{Im}(f)\sin\omega t$$

$$\langle x \rangle (t) = \sqrt{\frac{\hbar}{2 m \omega}}\, 2[(\mathrm{Re}(f) \cos \omega t + \mathrm{Im}(f) \sin \omega t].$$

$$\langle p \rangle (t) = \langle f \mid p(t) \mid f \rangle = \langle f \mid -m \omega x(0) \sin \omega t + p(0) \cos \omega t \mid f \rangle$$
$$= -m \omega \sqrt{\frac{\hbar}{2 m \omega}}\, 2 \,\mathrm{Re}(f) \sin\omega t + \sqrt{\frac{\hbar m \omega}{2}}\, 2 \,\mathrm{Im}(f) \cos \omega t$$

$$\langle p \rangle (t) = \sqrt{\frac{\hbar m \omega}{2}}\, 2[-\mathrm{Re}(f) \sin\omega t + \mathrm{Im}(f) \cos \omega t].$$

Indeed, these solutions correspond to the classical ones.

(i) [optional]

The Schrödinger equation gives
$$\mid n, t \rangle = e^{-i H t/\hbar} \mid n \rangle = e^{-i \hbar \omega (n+1/2) t/\hbar} \mid n \rangle = e^{-i \omega (n+1/2) t} \mid n \rangle.$$
Using what we showed above,
$$\mid f \rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle,$$
its time evolution is
$$\mid f, t \rangle = e^{-i H t/\hbar} \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle e^{-i \omega (n+1/2) t}$$
$$= \sum_{n=0}^{\infty} \frac{(f e^{-i \omega t})^n}{\sqrt{n!}} \left| n \right\rangle e^{-i \omega t/2} = \mid f e^{-i \omega t} \rangle e^{-i \omega t/2}$$

$$\mid f, t \rangle = \mid f e^{-i \omega t} \rangle e^{-i \omega t/2}.$$

where the coherent state in the last expression has the eigenvalue $a \mid f e^{-i \omega t} \rangle = f e^{-i \omega t} \mid f e^{-i \omega t} \rangle$.

(j)

The probability density on x is $\mid \langle x \mid f e^{-i \omega t} \rangle \mid^2$, i.e.

```
ρf[x_, t_] = Conjugate[ψf[x]] ψf[x] /. {f → Sqrt[m ω/(2 ℏ)] x0 Exp[-I ω t]};
```

. For the convenience in plotting, we set all the constants to 1

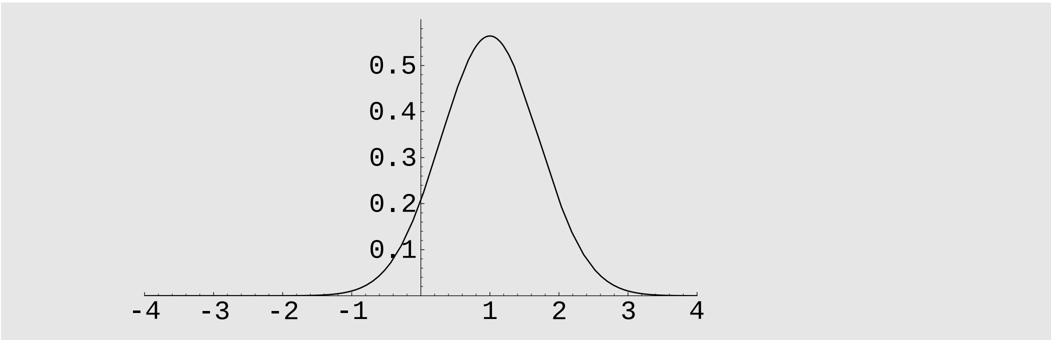
```
ρfc[x_, t_] = Simplify[ComplexExpand[ρf[x, t]] /. {m → 1, ℏ → 1, ω → 1, x0 → 1}]
```

$$\frac{e^{-(x - \mathrm{Cos}[t])^2}}{\sqrt{\pi}}$$

. Now we animate over time interval $t = (0,\ 2\pi)$ with gaps $\Delta t = \pi/4$:

```
<< Graphics`Animation`;

MoviePlot[ρfc[x, t], {x, -4, 4}, {t, 0, 2 π, π / 4},
  AspectRatio → .5, PlotRange → {{-4, 4}, {0, 0.6}}];
```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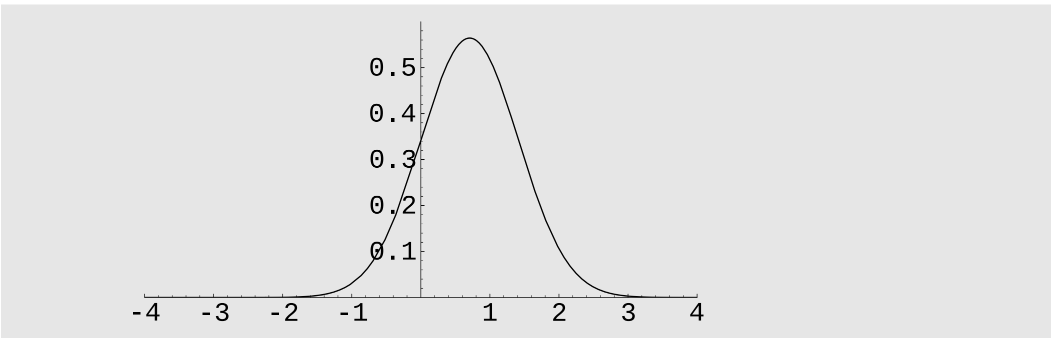
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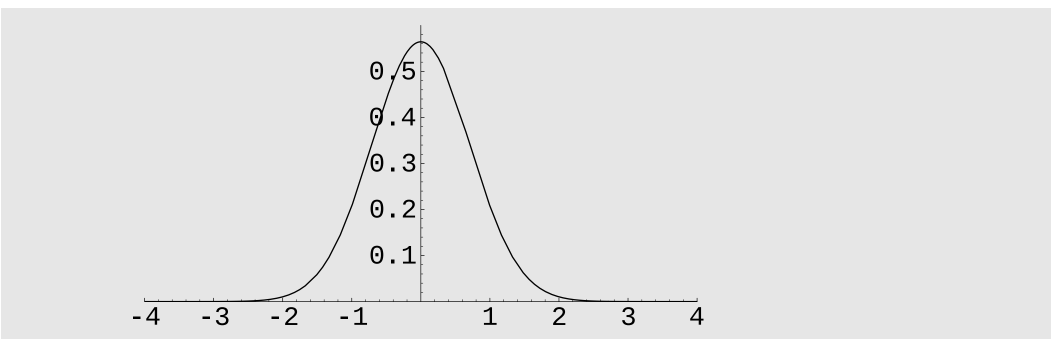
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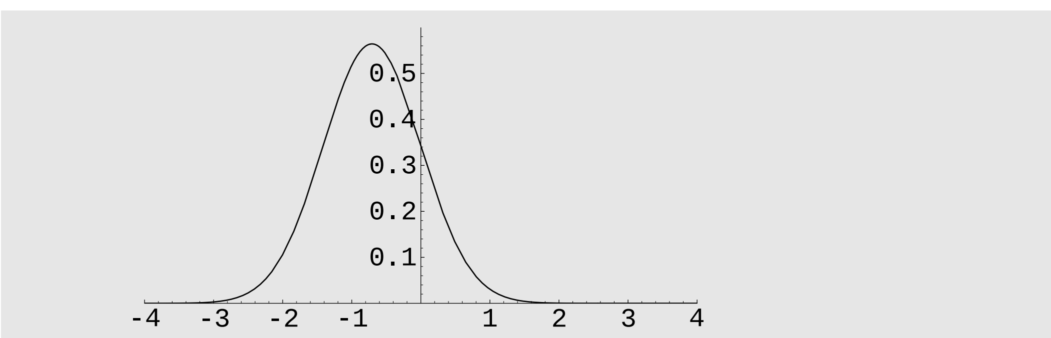
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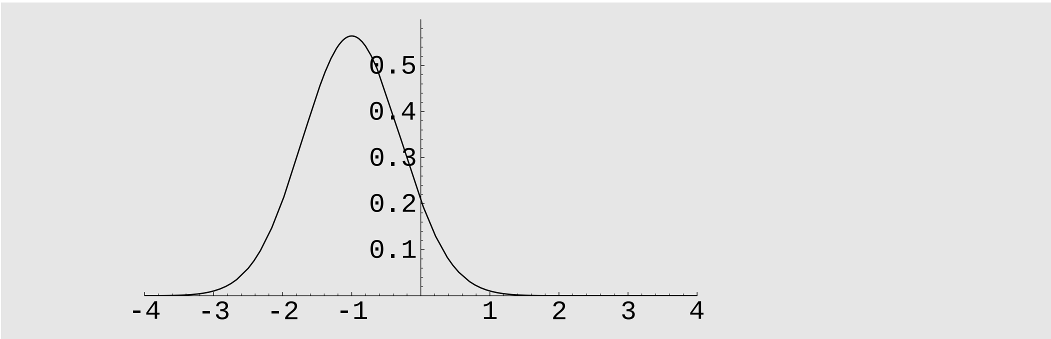
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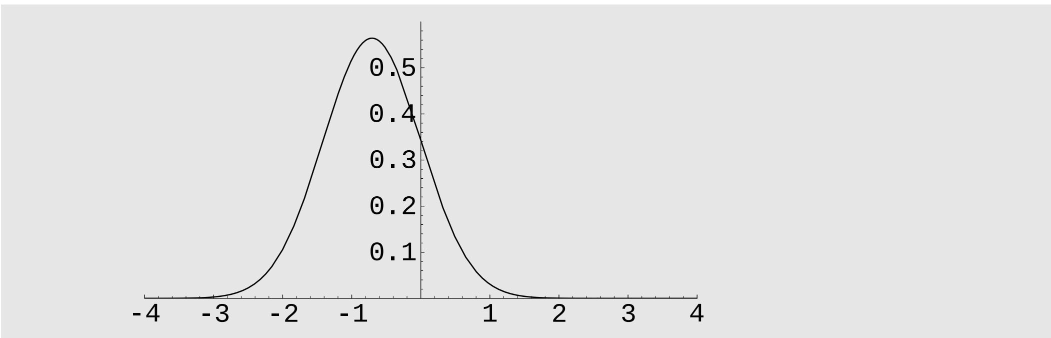
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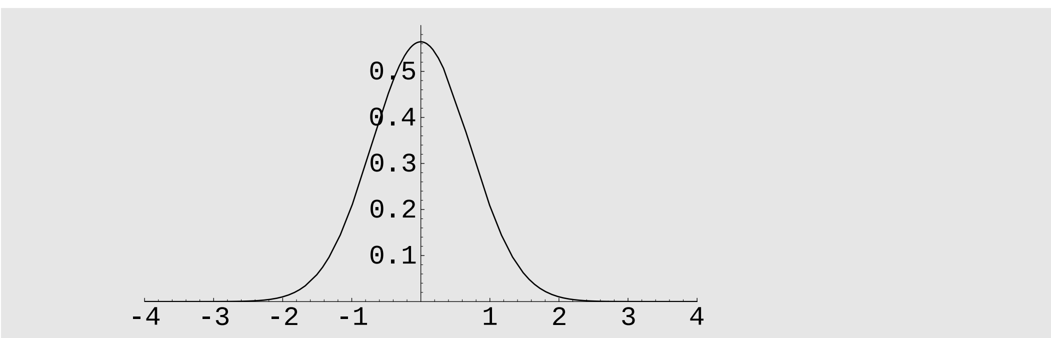
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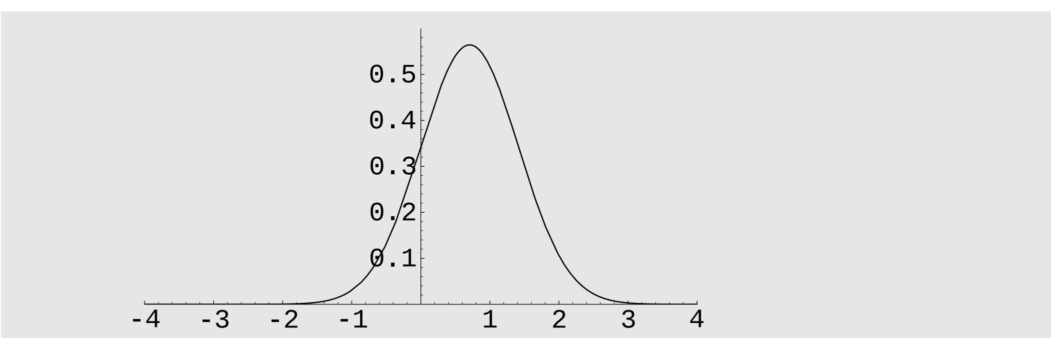
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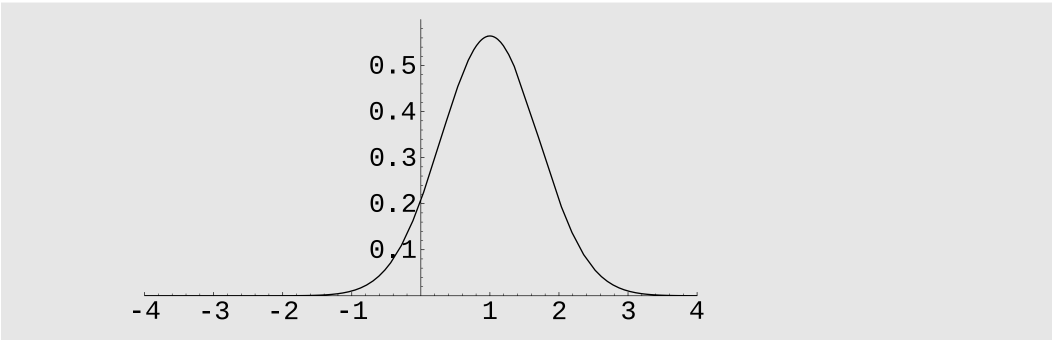
0.1
-4 -3 -2 -1 1 2 3 4



Perfectly oscillatory Gaussian! Here $m \sim \hbar$; however, as one would expect by the Correspondence Principle, as $m \gg \hbar$, the Gaussian approaches a delta distribution.

2. Time Evolution of Gaussian Wave Packet [optional]

(a)

The result for $\langle q \mid \psi \rangle = \phi(q)$ from Hwk 4 is

$$\phi[q\_] = \left(\frac{2\,d^2}{\pi\,\hbar^2}\right)^{1/4} \text{Exp}\left[\frac{I\,(p-q)\,x0}{\hbar}\right] \text{Exp}\left[\frac{-(p-q)^2\,d^2}{\hbar^2}\right];$$

. It is given that

$$\mid \psi, t > \; = \int e^{-i q^2\, t/2 m \hbar} \mid q \rangle \langle q \mid \psi \rangle \, dq$$

so

$$\psi(x, t) = \langle x \mid \psi, t \rangle = \int e^{-i q^2\, t/2 m \hbar} \langle x \mid q \rangle \langle q \mid \psi \rangle \, dq$$

$$= \int e^{-i q^2\, t/2 m \hbar} \frac{e^{i q x/\hbar}}{\sqrt{2 \pi \hbar}} \phi(q) \, dq \; .$$

This gives

```
myassumptions = {m > 0, ℏ > 0, d > 0, x0 > 0, {p, q, t} ∈ Reals};
ψ[x_, t_] =
 Integrate[Exp[-(I q^2 t)/(2 m ℏ)] 1/Sqrt[2 π ℏ] Exp[(I q x)/ℏ] ϕ[q], {q, -∞, ∞}, Assumptions → myassumptions]
```

$$\frac{e^{\frac{i\left(2 i d^2 p (p t-2 m x)+\hbar\left(m (x-x0)^2+2 p t x0\right)\right)}{2 \hbar \left(-2 i d^2 m+\hbar t\right)}} \left(\frac{2}{\pi}\right)^{1/4}}{\sqrt{2 d+\frac{i \hbar t}{d m}}}$$

(Note that I defined ’myassumptions’ for use later so *Mathematica* understands the nature of the constants here.)

This result looks a little odd with the i’s floating about and t in the denominators, but $\psi(x, 0)$ is indeed just the $\psi(x)$ given in Hwk 4:

```
TrigToExp[Simplify[ComplexExpand[ψ[x, 0]], Assumptions → myassumptions]]
```

$$\frac{e^{\frac{i p x}{\hbar}-\frac{(x-x0)^2}{4 d^2}}}{\sqrt{d}\,(2 \pi)^{1/4}}$$

(b)

As we know

$$\langle x\rangle(t) = \langle\psi, t \mid x \mid \psi, t\rangle = \int \psi(x,t)^* \, x \, \psi(x,t) \, dx$$

which gives

```
meanx = Integrate[ComplexExpand[Conjugate[ψ[x, t]] x ψ[x, t]],
  {x, -∞, ∞}, Assumptions → myassumptions]
```

$$\frac{p\,t}{m} + x0$$

This is the expected result!

(c)

The second moment

$$\langle x^2\rangle(t) = \langle\psi, t \mid x^2 \mid \psi, t\rangle = \int \psi(x,t)^* \, x^2 \, \psi(x,t) \, dx$$

which gives

```
meanx2 = Integrate[ComplexExpand[Conjugate[ψ[x, t]] x^2 ψ[x, t]],
  {x, -∞, ∞}, Assumptions → myassumptions]
```

$$\frac{4\,d^4\,m^2 + \hbar^2\,t^2 + 4\,d^2\,(p\,t + m\,x0)^2}{4\,d^2\,m^2}$$

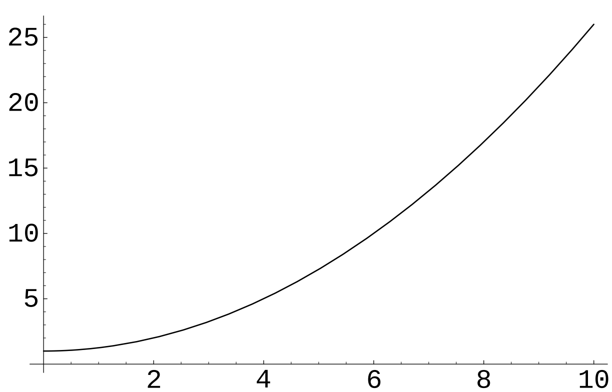
So the dispersion−square $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ is

```
dispx2 = Simplify[meanx2 - meanx^2, Assumptions → myassumptions]
```

$$d^2 + \frac{\hbar^2\,t^2}{4\,d^2\,m^2}$$

Again, as expected, the dispersion increases with time; specifically, the disperson−square increases quadratically, as shown below with all constants set to 1.

```
Plot[dispx2 /. {d → 1, ℏ → 1, m → 1}, {t, 0, 10}];
```



(d)

There is only one Heisenberg equation of motion since $[p, H] = 0$, so like in part 1(h)

$$i\hbar \frac{d}{dt} x = [x, H] = \left[x, \frac{p^2}{2m}\right] = i\hbar \frac{p}{m},$$

which clearly gives

$$x(t) = x(0) + \frac{p}{m} t$$

or

$$x(t) = x + \frac{t}{m} \frac{\hbar}{i} \frac{\partial}{\partial x}$$

on the x basis.

For the second moment,

$$\frac{d}{dt} x^2 = x \frac{dx}{dt} + \frac{dx}{dt} x = (x(0) + \frac{p}{m} t) \frac{p}{m} + \frac{p}{m} (x(0) + \frac{p}{m} t)$$

so

$$x^2(t) = (\frac{p}{m})^2 t^2 + (x(0) \frac{p}{m} + \frac{p}{m} x(0))\, t + x^2(0)$$

or

$$x^2(t) = \left(\frac{t}{m} \frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial x^2} + \frac{t}{m} \frac{\hbar}{i} \left(2\, x \frac{\partial}{\partial x} + 1\right) + x^2 .$$

(e)

Again like in part 1(h),

$$\langle x \rangle (t) = \langle \psi \mid x(t) \mid \psi \rangle = \int \psi(x, 0)^* \, x(t)\, \psi(x, 0)\, dx$$

which gives

```
meanxh = Integrate[ComplexExpand[Conjugate[ψ[x, 0]] (x ψ[x, 0] + (t/m) (ħ/I) ∂x ψ[x, 0])],
   {x, -∞, ∞}, Assumptions → myassumptions]
```

$$\frac{p\, t}{m} + x0$$

Similarly, for $\langle x^2 \rangle (t)$

```
meanx2h = Integrate[ComplexExpand[
    Conjugate[ψ[x, 0]] (((t/m) (ħ/I))^2 ∂x ∂x ψ[x, 0] + (t/m) (ħ/I) (2 x ∂x ψ[x, 0] + ψ[x, 0]) + x^2 ψ[x, 0])],
   {x, -∞, ∞}, Assumptions → myassumptions]
```

$$\frac{4\, d^4\, m^2 + \hbar^2\, t^2 + 4\, d^2\, (p\, t + m\, x0)^2}{4\, d^2\, m^2}$$

. Thus, for $\left\langle (\Delta\, x)^2 \right\rangle (t)$ we find

```
dispx2h = Simplify[meanx2h - meanxh^2, Assumptions → myassumptions]
```

$$d^2 + \frac{\hbar^2 t^2}{4 d^2 m^2}$$

These results in the Heisenberg picture are just as we calculated in parts (b) and (c) for the Schroedinger picture!

To check:

```
meanx - meanxh
dispx2 - dispx2h
```

0

0