# HW #3 (221A), due Sep 15, 4pm

1. Answer the following questions about the spin operators using their matrix representation,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

   (a) Show the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$.

   (b) Construct the eigenstates of the spin operator $S_{\vec{n}} = \vec{n} \cdot \vec{S}$ along an arbitrary axis $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and show that eigenvalues are again $\pm\frac{\hbar}{2}$.

   (c) Calculate the probability that the state with $S_{\vec{n}} = +\frac{\hbar}{2}$ would be measured to have $S_z = +\frac{\hbar}{2}$.

   (d) Generalize the result to the spin along the $\vec{n}$ direction measured along the $\vec{n}'$ direction, and show that the result depends only on the angle between $\vec{n}$ and $\vec{n}'$ as expected from the rotational invariance.

2. Here is a sloppy way to estimate the energy of the hydrogen atom using the uncertainty principle. Using the energy

$$E = \frac{\vec{p}^2}{2m} - \frac{Ze^2}{r}, \tag{2}$$

   and assuming that the "size" ($\Delta x$) of the wave function is approximately $d$, the typical order of magnitude of the momentum is given by $p = \Delta p \simeq \hbar/d$. (Here I assumed that the momentum distribution is peaked at zero symmetrically, so that I can equate $p$ and $\Delta p$). The typical size of the potential energy is then $Ze^2/d$. Minimize the energy with respect to $d$ and find the "ground state energy." Compare the estimate to the exact result.