HW #3

1. Spin Matrices

We use the spin operators represented in the bases where S_z is diagonal:

```
Sx = ℏ/2 {{0, 1}, {1, 0}}; Sy = ℏ/2 {{0, -I}, {I, 0}}; Sz = ℏ/2 {{1, 0}, {0, -1}};
```

```
Sx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

```
Sy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}$$

```
Sz // MatrixForm
```

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

(a) Obviously two matrices commute when they are the same: $i = j$. Also, it is obvious that $[S_i, S_j]$ is anti−symmetric in $i \longleftrightarrow j$ because $[S_j, S_i] = -[S_i, S_j]$. Therefore, it only reamains to verify

```
Sx.Sy - Sy.Sx - I ℏ Sz
```

```
{{0, 0}, {0, 0}}
```

```
Sy.Sz - Sz.Sy - I ℏ Sx
```

```
{{0, 0}, {0, 0}}
```

```
Sz.Sx - Sx.Sz - I ℏ Sy
```

```
{{0, 0}, {0, 0}}
```

(b) We define $\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$

```
nx = Sin[θ] Cos[ϕ]; ny = Sin[θ] Sin[ϕ]; nz = Cos[θ]
```

```
Cos[θ]
```

```
Sn = Simplify[nx Sx + ny Sy + nz Sz]
```

$$\left\{\left\{\frac{1}{2}\hbar\,\text{Cos}[\theta],\ \frac{1}{2}\hbar\,\text{Sin}[\theta]\,(\text{Cos}[\phi]-i\,\text{Sin}[\phi])\right\},\ \left\{\frac{1}{2}\hbar\,\text{Sin}[\theta]\,(\text{Cos}[\phi]+i\,\text{Sin}[\phi]),\ -\frac{1}{2}\hbar\,\text{Cos}[\theta]\right\}\right\}$$

```
Eigensystem[Sn]
```

$$\left\{\left\{-\frac{\sqrt{\hbar^2}}{2},\ \frac{\sqrt{\hbar^2}}{2}\right\},\ \left\{\left\{\frac{\left(-\sqrt{\hbar^2}+\hbar\,\text{Cos}[\theta]\right)\text{Csc}[\theta]}{\hbar\,(\text{Cos}[\phi]+i\,\text{Sin}[\phi])},\ 1\right\},\ \left\{\frac{\left(\sqrt{\hbar^2}+\hbar\,\text{Cos}[\theta]\right)\text{Csc}[\theta]}{\hbar\,(\text{Cos}[\phi]+i\,\text{Sin}[\phi])},\ 1\right\}\right\}\right\}$$

```
PowerExpand[%]
```

$$\left\{\left\{-\frac{\hbar}{2},\ \frac{\hbar}{2}\right\},\ \left\{\left\{\frac{(-\hbar+\hbar\,\text{Cos}[\theta])\,\text{Csc}[\theta]}{\hbar\,(\text{Cos}[\phi]+i\,\text{Sin}[\phi])},\ 1\right\},\ \left\{\frac{(\hbar+\hbar\,\text{Cos}[\theta])\,\text{Csc}[\theta]}{\hbar\,(\text{Cos}[\phi]+i\,\text{Sin}[\phi])},\ 1\right\}\right\}\right\}$$

```
Simplify[%]
```

$$\left\{\left\{-\frac{\hbar}{2},\ \frac{\hbar}{2}\right\},\ \left\{\left\{(-\text{Cos}[\phi]+i\,\text{Sin}[\phi])\,\text{Tan}\left[\frac{\theta}{2}\right],\ 1\right\},\ \left\{\text{Cot}\left[\frac{\theta}{2}\right](\text{Cos}[\phi]-i\,\text{Sin}[\phi]),\ 1\right\}\right\}\right\}$$

Therefore, one can take the the normalized eigenstates to be $\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$ with eigenvalue $+\frac{\hbar}{2}$ and $\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$ with eigenvalue $-\frac{\hbar}{2}$.

(c) The state with spin along the $\vec{n}$ direction is the former, and its probability to have the positive S_z when measured is simply given by $\left|\left\langle S_z=+\frac{\hbar}{2}\,\middle|\,S_n=+\frac{\hbar}{2}\right\rangle\right|^2=\left|(1,0)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}\right|^2=\cos^2\frac{\theta}{2}$.

(d) Between $\vec{n}=(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ and $\vec{n}'=(\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta')$, the probability is $\left|\left\langle S_{n'}=+\frac{\hbar}{2}\,\middle|\,S_n=+\frac{\hbar}{2}\right\rangle\right|^2=\left|\left(\cos\frac{\theta'}{2},\sin\frac{\theta'}{2}e^{-i\phi'}\right)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}\right|^2=\left|\cos\frac{\theta'}{2}\cos\frac{\theta}{2}+\sin\frac{\theta'}{2}e^{-i\phi'}\sin\frac{\theta}{2}e^{i\phi}\right|^2=$

$$\cos^2\frac{\theta'}{2}\cos^2\frac{\theta}{2}+\sin^2\frac{\theta'}{2}\sin^2\frac{\theta}{2}+2\cos\frac{\theta'}{2}\cos\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\cos(\phi-\phi')$$

```
TrigExpand[Cos[θ1/2]^2 Cos[θ2/2]^2 + Sin[θ1/2]^2 Sin[θ2/2]^2 +
  2 Cos[θ1/2] Cos[θ2/2] Sin[θ1/2] Sin[θ2/2] Cos[ϕ1 - ϕ2]]
```

$$\frac{1}{2}+\frac{1}{2}\text{Cos}\left[\frac{\theta_1}{2}\right]^2\text{Cos}\left[\frac{\theta_2}{2}\right]^2-\frac{1}{2}\text{Cos}\left[\frac{\theta_2}{2}\right]^2\text{Sin}\left[\frac{\theta_1}{2}\right]^2+2\,\text{Cos}\left[\frac{\theta_1}{2}\right]\text{Cos}\left[\frac{\theta_2}{2}\right]\text{Cos}[\phi_1]\,\text{Cos}[\phi_2]\,\text{Sin}\left[\frac{\theta_1}{2}\right]\text{Sin}\left[\frac{\theta_2}{2}\right]-\frac{1}{2}\text{Cos}\left[\frac{\theta_1}{2}\right]^2\text{Sin}\left[\frac{\theta_2}{2}\right]^2+\frac{1}{2}\text{Sin}\left[\frac{\theta_1}{2}\right]^2\text{Sin}\left[\frac{\theta_2}{2}\right]^2+2\,\text{Cos}\left[\frac{\theta_1}{2}\right]\text{Cos}\left[\frac{\theta_2}{2}\right]\text{Sin}\left[\frac{\theta_1}{2}\right]\text{Sin}\left[\frac{\theta_2}{2}\right]\text{Sin}[\phi_1]\,\text{Sin}[\phi_2]$$

```
Simplify[%]
```

$$\frac{1}{2}\left(1+\text{Cos}[\theta_1]\,\text{Cos}[\theta_2]+\text{Cos}[\phi_1]\,\text{Cos}[\phi_2]\,\text{Sin}[\theta_1]\,\text{Sin}[\theta_2]+\text{Sin}[\theta_1]\,\text{Sin}[\theta_2]\,\text{Sin}[\phi_1]\,\text{Sin}[\phi_2]\right)$$

This is nothing but $\frac{1}{2}\left(1+\vec{n}\cdot\vec{n}'\right)=\frac{1}{2}(1+\cos\eta)=\cos^2\frac{\eta}{2}$, where η is the angle between two vectors, as expected from the rotational invariance.

2. Sloppy Hydrogen Atom

According to the problem,

$$\text{Energy} = \frac{1}{2\,m}\left(\frac{\hbar}{d}\right)^2 - \frac{Z\,e^2}{d}$$

$$-\frac{e^2\,Z}{d}+\frac{\hbar^2}{2\,d^2\,m}$$

```
Solve[D[Energy, d] == 0, d]
```

$$\left\{\left\{d\to\frac{\hbar^2}{e^2\,m\,Z}\right\}\right\}$$

```
Simplify[Energy /. %[[1]]]
```

$$-\frac{e^4\,m\,Z^2}{2\,\hbar^2}$$

This actually agrees with the exact result. (One should be cautioned, however, that the agreement with the exact result is a coincidence for this particular example.)