221A HW #2, due Sep 8 (Fri), 4pm

1. Free Particle Schrödinger Equation

Time-independent Schrödinger equation for a free particle is given by

$$\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial \vec{x}}\right)^2 \psi(\vec{x}) = E\psi(\vec{x}). \tag{1}$$

It is customary to write $E = \hbar^2 k^2/2m$ to simplify the equation

$$(\vec{\nabla}^2 + k^2)\psi(\vec{x}) = 0. \tag{2}$$

Show that (a) a plane wave $\psi(\vec{x}) = e^{ikz}$, and (b) a spherical wave $\psi(\vec{x}) = e^{ikr}/r$ ($r = \sqrt{x^2+y^2+z^2}$) satisfy the equation. (In either case, the wavelength of the solution is given by $\lambda = 2\pi/k$ and the momentum by de Broglie's relation $p = \hbar k$.)

2. Double Pinhole Experiment

Besides Stern–Gerlach experiment we discussed in the class, double-slit experiment is another one that demonstrates how different quantum mechanics is from the classic counterpart. To avoid mathematical complications with Bessel function, we discuss two pinholes rather than slits.

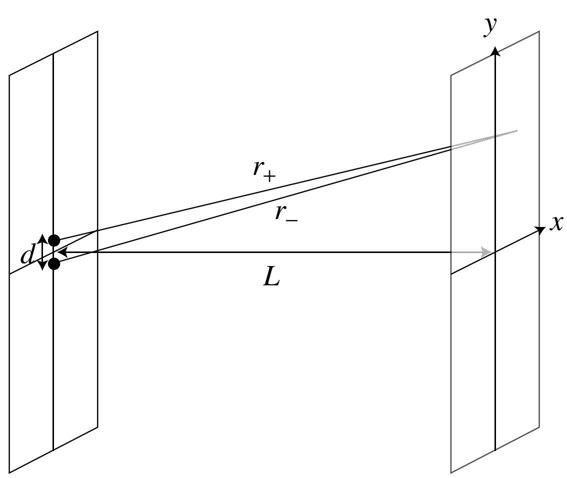

Figure 1: The double pinhole experiment.

Suppose you send in an electron along the z-axis on a screen at $z = 0$ with two pinholes at $x = 0$, $y = \pm d/2$. On a point (x, y) on another screen at $z = L \gg d, \lambda$, the distance from each pinhole is given by $r_\pm = \sqrt{x^2 + (y \mp d/2)^2 + L^2}$.

The spherical wave from each pinhole is added on the screen and hence the wave function is

$$\psi(x, y) = \frac{e^{ikr_+}}{r_+} + \frac{e^{ikr_-}}{r_-}, \tag{3}$$

where $k = 2\pi/\lambda$. Answer the following questions.

(a) Considering just the exponential factors, show that the constructive interference appears approximately at

$$\frac{y}{r} = n\frac{\lambda}{d} \qquad (n \in \mathbb{Z}). \tag{4}$$

where $r = \sqrt{x^2 + y^2 + L^2}$.

(b) Make a plot of the intensity $|\psi(0, y)|^2$ as a function of y, by choosing $k = 1$, $d = 20$, and $L = 1000$. Use Mathematica `Plot` function. The intensity $|\psi(0, y)|^2$ is interpreted as the probability distribution for the electron to be detected on the screen, after repeating the same experiment many many times.

(c) Make a contour plot of the intensity $|\psi(x, y)|^2$ as a function of x and y, for the same parameters, using Mathematica `ContourPlot` function.

(d) If you place a counter at both pinholes to see if the electron has passed one of them, all of a sudden the wave function "collapses." If the electron is observed to pass through the pinhole at $y = +d/2$, the wave function becomes

$$\psi_+(x, y) = \frac{e^{ikr_+}}{r_+}. \tag{5}$$

If it is observed to pass through that at $y = -d/2$, the wave function becomes

$$\psi_-(x, y) = \frac{e^{ikr_-}}{r_-}. \tag{6}$$

After repeating this experiment many times with 50:50 probability for each of the pinholes, the probability on the screen will be given by

$$|\psi_+(x, y)|^2 + |\psi_-(x, y)|^2 \tag{7}$$

instead. Plot this function on y-axis, and also show the contour plot, to compare its pattern to the case when you do not place a counter. What is the difference from the case without the counter?