Dirac Delta Function

1 Definition

Dirac’s delta function is defined by the following property

\[ \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \]

with

\[ \int_{t_1}^{t_2} dt \delta(t) = 1 \]

if \(0 \in [t_1, t_2]\) (and zero otherwise). It is “infinitely peaked” at \(t = 0\) with the total area of unity. You can view this function as a limit of Gaussian

\[ \delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi} \sigma} e^{-t^2/(2\sigma^2)} \]

or a Lorentzian

\[ \delta(t) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}. \]

The important property of the delta function is the following relation

\[ \int dt f(t) \delta(t) = f(0) \]

for any function \(f(t)\). This is easy to see. First of all, \(\delta(t)\) vanishes everywhere except \(t = 0\). Therefore, it does not matter what values the function \(f(t)\) takes except at \(t = 0\). You can then say \(f(t)\delta(t) = f(0)\delta(t)\). Then \(f(0)\) can be pulled outside the integral because it does not depend on \(t\), and you obtain the r.h.s. This equation can easily be generalized to

\[ \int dt f(t) \delta(t - t_0) = f(t_0). \]

Mathematically, the delta function is not a function, because it is too singular. Instead, it is said to be a “distribution.” It is a generalized idea of functions, but can be used only inside integrals. In fact, \(\int dt \delta(t)\) can be regarded as an “operator” which pulls the value of a function at zero. Put it this way, it sounds perfectly legitimate and well-defined. But as long as it is understood that the delta function is eventually integrated, we can use it as
if it is a function. One caveat is that you are not allowed to multiply delta functions whose arguments become simultaneously zero, e.g., $\delta(t)^2$. If you try to integrate it over $t$, you would obtain $\delta(0)$, which is infinite and does not make sense. But physicists are sloppy enough to even use $\delta(0)$ sometimes, as we will discuss below.

## 2 Fourier Transformation

It is often useful to talk about Fourier transformation of functions. For a function $f(t)$, you define its Fourier transform

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} dt \frac{e^{its}}{\sqrt{2\pi}} f(t).$$

This transform is reversible, i.e., you can go back from $\tilde{f}(s)$ to $f(t)$ by

$$f(t) = \int_{-\infty}^{\infty} ds \frac{e^{-its}}{\sqrt{2\pi}} \tilde{f}(s).$$

You may recall that the patterns from optical or X-ray diffraction are Fourier transforms of the structure. For example, Laue determined the crystallographic structure of solid by doing inverse Fourier-transform of the X-ray diffraction patterns.

If you set $f(t) = \delta(t)$ in the above equations, you find

$$\tilde{\delta}(s) \equiv \int_{-\infty}^{\infty} dt \frac{e^{its}}{\sqrt{2\pi}} \delta(t) = \frac{1}{\sqrt{2\pi}},$$

$$\delta(t) = \int_{-\infty}^{\infty} ds \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} ds \frac{e^{-its}}{2\pi}.$$  

In other words, the delta function and a constant $1/\sqrt{2\pi}$ are Fourier-transform of each other.

Another way to see the integral representation of the delta function is again using the limits. For example, using the limit of the Gaussian Eq. (3),

$$\delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi} \sigma} e^{-t^2/2\sigma^2}$$

$$= \lim_{\sigma \to 0} \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} e^{-\omega^2/2\sigma^2} e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{2\pi}.$$  


3 Position Space

Dirac invented the delta function to deal with the completeness relation for position and momentum eigenstates. The eigenstate for the position operator \( x \)

\[
x | x' \rangle = x' | x' \rangle
\]  

(12)
must be normalized in a way that the analogue of the completeness relation holds for discrete eigenstates \( 1 = \sum |a\rangle\langle a| \). Because the eigenvalues of the position operator are continuous, the sum is replaced by an integral

\[
1 = \int |x'\rangle dx' \langle x' |.
\]  

(13)

For the case of the discrete eigenstates, using the completeness relationship twice gives a consistent result because of the orthonormality of the eigenstates \( \langle a'|a'' \rangle = \delta_{a',a''} \):

\[
1 = 1 \times 1 = \left( \sum_{a'} |a'\rangle\langle a'| \right) \left( \sum_{a''} |a''\rangle\langle a'' | \right)
\]

\[
= \sum_{a',a''} |a'\rangle\langle a'| \langle a''|\langle a'' \rangle
\]

\[
= \sum_{a',a''} |a'\rangle \delta_{a',a''} \langle a''|\langle a'' \rangle
\]

\[
= \sum_{a'} |a'\rangle\langle a'| = 1.
\]  

(14)

Therefore, we need also the states \( |x'\rangle \) to be orthonormal. To see it, we try the same thing as in the discrete spectrum

\[
1 = 1 \times 1 = \left( \int |x'\rangle dx' \langle x' | \right) \left( \int |x''\rangle dx'' \langle x'' | \right)
\]

\[
= \int dx' dx'' |x'\rangle \langle x'| \langle x''|\langle x'' \rangle.
\]  

(15)

Now we can determine what the “orthonormality” condition must look like. Only by setting \( \langle x'|x'' \rangle = \delta(x' - x'') \), we find

\[
1 = \int dx' dx'' |x'\rangle \delta(x' - x'') \langle x''|\langle x'' \rangle
\]

\[
= \int dx' |x'\rangle \langle x'| = 1.
\]  

(16)
At the last step, I used the property of the delta function that the integral over $x''$ inserts the value $x'' = x'$ into the rest of the integrand. This is why we need the “delta-function normalization” for the position eigenkets.

It is also worthwhile to note that the delta function in position has the dimension of $1/L$, because its integral over the position is unity. Therefore the position eigenket $|x'angle$ has the dimension of $L^{-1/2}$.

4 Momentum Space

As you see in Sakurai Eq. (1.7.32), the eigenstates of the position and momentum operators have the inner product

$$\langle x' | p' \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i p' x'/\hbar} \tag{17}$$

From this expression, you can see that the wave functions in the position space and the momentum space are related by the Fourier-transform.

$$\phi_{\alpha}(p') = \langle p'|\alpha \rangle = \int \langle p'|x' \rangle dx' \langle x'|\alpha \rangle = \int dx' e^{-i p' x'/\hbar} \frac{1}{\sqrt{2\pi\hbar}} \psi_{\alpha}(x'). \tag{18}$$

The completeness of the momentum eigenstates can also be shown using the properties of the delta function.

$$\int |p'angle dp' \langle p'| = \int dp' dx' dx'' |x'\rangle \langle x'|p'\rangle \langle p'|x''\rangle \langle x''|$$

$$= \int dp' dx' dx'' |x'\rangle \frac{e^{i x' p'/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{-i x'' p'/\hbar}}{\sqrt{2\pi\hbar}} |x''|$$

$$= \int dx' dx'' |x'\rangle \langle x''| \int dp' \frac{e^{i(x'-x'') p'/\hbar}}{2\pi\hbar}. \tag{19}$$

The last integral, after changing the variable from $p'$ to $k = p/\hbar$, is nothing but the Fourier-integral expression for the delta function. Therefore,

$$= \int dx' dx'' |x'\rangle \langle x''| \delta(x' - x'')$$

$$= \int dx' |x'\rangle \langle x'| = 1. \tag{20}$$

This proves the completeness of the momentum eigenstates.