

# 221A Miscellaneous Notes

## Continuity Equation

### 1 The Continuity Equation

As I received questions about the midterm problems, I realized that some of you have a conceptual gap about the continuity equation. It appears in Sakurai, pp. 101–102, but he does not go into the general discussions about what is meant by the one of the most famous equations in physics (Sakurai (2.4.15)),

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \quad (1)$$

called the continuity equation. Let me review what it means.

Imagine you contain the system completely in a volume, so that nothing is going in and out at the surface. Integrate the both sides of the equation over this entire volume.

$$\int_V dV \left[ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} \right] = 0. \quad (2)$$

Using the Gauss' theorem, the second term can be rewritten as a surface integral,

$$\int_V dV \vec{\nabla} \cdot \vec{j} = \int_{\partial V} d\vec{S} \cdot \vec{j}, \quad (3)$$

where the surface element  $d\vec{S}$  is defined outwards and integrated over the surface  $\partial V$  of the volume  $V$ . In our case, because we contained the system completely in the volume, there is nothing at the surface and this term vanishes. For the first term, interchanging the order of the volume integral and the time derivative, it becomes

$$\frac{\partial}{\partial t} \int_V dV \rho \quad (4)$$

Therefore, the equation is

$$\frac{\partial}{\partial t} \int_V dV \rho = 0, \quad (5)$$

and hence

$$\int_V dV \rho \quad (6)$$

is *conserved*.

In fact, even if the volume does not contain the full system, the amount of the change in  $\int dV \rho$  is given by the amount of flow out from the volume,

$$\frac{\partial}{\partial t} \int dV \rho = - \int d\vec{S} \cdot \vec{j}, \quad (7)$$

which gives the interpretation that the  $\vec{j}(\vec{x}, t)$  is the *current* of the conserved quantity, while  $\rho(\vec{x}, t)$  is the *density* of the conserved quantity.

## 2 Examples

This equation shows up in many many different contexts in physics.

### 2.1 Fluid Mechanics

Probably the first one is in fluid mechanics. A fluid is described by the mass density  $\rho(\vec{x}, t)$  and its mass flow (or momentum density if you like)  $\vec{j}(\vec{x}, t) = \rho(\vec{x}, t) \vec{v}(\vec{x}, t)$ , where  $\vec{v}(\vec{x}, t)$  is the velocity field. The conserved quantity is the total mass of the fluid,

$$M = \int_V dV \rho. \quad (8)$$

### 2.2 Heat Conduction

In the problem of heat conduction, we solve the equation

$$\frac{\partial T}{\partial t} = \kappa \Delta T. \quad (9)$$

This equation is nothing but the continuity equation for the thermal energy (divided by its heat capacity)  $\rho = T$  with an additional idea that the heat flux vector or heat flow goes along the slope of the temperature field:  $\vec{j} = \kappa \vec{\nabla} T$ .

### 2.3 Diffusion

The heat conduction equation is just one example of diffusion equations. For example, the diffusion of gas is described by

$$\frac{\partial n}{\partial t} = D \Delta n, \quad (10)$$

where  $n$  is the number density,  $D$  the diffusion constant. The conserved quantity is the total number of gas molecules

$$N = \int dV n, \quad (11)$$

and the current is  $\vec{j} = D\vec{\nabla}n$ .

## 2.4 Electromagnetism

Another example is in the electromagnetism. Look at the Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho, \quad (12)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \dot{\vec{E}} + \mu_0 \vec{j}, \quad (13)$$

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}, \quad (14)$$

$$\vec{\nabla} \cdot \vec{B} = 0. \quad (15)$$

The electric charge density  $\rho$  and the electric current density  $\vec{j}$  appear. By taking the time derivative of the first equation, we find

$$\vec{\nabla} \cdot \dot{\vec{E}} = \frac{1}{\epsilon_0} \dot{\rho}, \quad (16)$$

while the divergence of the second equation is

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \frac{1}{c^2} \vec{\nabla} \cdot \dot{\vec{E}} + \mu_0 \vec{\nabla} \cdot \vec{j}. \quad (17)$$

Note that  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$ , and hence the second equation becomes

$$\vec{\nabla} \cdot \dot{\vec{E}} = -c^2 \mu_0 \vec{\nabla} \cdot \vec{j}. \quad (18)$$

Comparing Eqs. (16,18), we find that the Maxwell's equations are consistent *only if*

$$\dot{\rho} = -c^2 \mu_0 \epsilon_0 \vec{\nabla} \cdot \vec{j}. \quad (19)$$

Of course,  $c^2 = 1/(\mu_0 \epsilon_0)$  and this is nothing but the continuity equation. The conserved quantity is the total electric charge,

$$Q = \int_V dV \rho. \quad (20)$$

The continuity equation can be written in a manifestly Lorentz-invariant fashion. The charge density and the current form a four-vector

$$j^\mu = (c\rho, \vec{j}), \quad (21)$$

and together with  $x^\mu = (ct, \vec{x})$  and hence  $\partial_\mu = (\frac{1}{c}\partial_t, \vec{\nabla})$ , the continuity equation becomes

$$\partial_\mu j^\mu = 0. \quad (22)$$

## 2.5 General Relativity

Einstein's equation is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi\frac{G_N}{c^4}T^{\mu\nu}, \quad (23)$$

where  $R^{\mu\nu}$  is the Ricci curvature tensor,  $g^{\mu\nu}$  the metric tensor, and  $R = g_{\mu\nu}R^{\mu\nu}$  the Ricci scalar curvature. The right-handed side of the equation is the source for the gravitational field  $T^{\mu\nu}$ , the energy-momentum tensor.

Similarly to the Maxwell's equations, the Einstein's equation is consistent only if the energy-momentum tensor is conserved,

$$\nabla_\mu T^{\mu\nu} = 0, \quad (24)$$

where  $\nabla_\mu$  is the covariant derivative, a generalization of the divergence to curved spacetime.

## 2.6 Quantum Mechanics

In quantum mechanics,  $\rho = \psi^*\psi$  is the probability density, namely the probability to find the particle in a given spatial volume element. The particle is supposed to be somewhere in the whole space, and hence the total probability

$$P = \int dV \rho = 1 \quad (25)$$

is 100%. If this is the case at a given moment, it should stay that way for the rest of the evolution; it is supposed to be somewhere in the whole space *anytime*. What it means is that the total probability is conserved, and hence  $\partial P/\partial t = 0$ . Another equivalent statement is that, once the wave function is

normalized, it stays normalized forever. This is a consequence of the unitarity of the time-evolution operator,

$$\langle \psi, t | \psi, t \rangle = \left( \langle \psi, 0 | e^{+iHt/\hbar} \right) \left( e^{-iHt/\hbar} | \psi, 0 \rangle \right) = \langle \psi, 0 | \psi, 0 \rangle. \quad (26)$$

It sounds funny to say that the probability is conserved, but it has to be. Correspondingly, there must be a probability current. What we are looking for is the current  $\vec{j}$  that satisfies the continuity equation Eq. (1) with  $\rho = \psi^* \psi$ . In the case of a free particle, together with the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi, \quad (27)$$

we find

$$\frac{\partial}{\partial t} \psi^* \psi = \psi^* \frac{i\hbar}{2m} \Delta \psi - \left( \frac{i\hbar}{2m} \Delta \psi^* \right) \psi = -\vec{\nabla} \cdot \vec{j}. \quad (28)$$

By inspection, we find

$$\vec{j} = \psi^* \frac{\hbar}{2mi} \vec{\nabla} \psi - \left( \frac{\hbar}{2mi} \vec{\nabla} \psi^* \right) \psi. \quad (29)$$

For the plane wave  $\psi = e^{i\vec{p}\cdot\vec{x}/\hbar} / (2\pi\hbar)^{3/2}$ , we find

$$\vec{j} = \frac{1}{(2\pi\hbar)^3} \frac{\vec{p}}{m} = \rho \frac{\vec{p}}{m}. \quad (30)$$

One important caveat is that this form of the probability current may not be true depending on the form of the Hamiltonian, *e.g.*, the presence of the vector potential, if the wave function has more than one particles, etc. The explicit form needs to be determined knowing what the system is.