HW #8 (221A), due Nov 5, 4pm

1. Consider a three-dimensional isotropic harmonic oscillator with Hamiltonian
$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2} m\omega^2 \vec{x}^2.$$
(This is the starting point of the shell model of nuclei.) Answer the following questions.

   (a) Clearly, the system is spherically symmetric, and hence there is a conserved angular momentum vector. Show that $\vec{L} = \vec{x} \times \vec{p}$ commutes with the Hamiltonian.

   (b) Define three sets of creation and annihilation operators a_i and $a_i^\dagger$ for $i = x, y, z$. Rewrite H and $\vec{L}$ in terms of creation and annihilation operators.

   (c) Show that $|0\rangle$ belongs to the $l = 0$ representation.

   (d) Show that the $N = 1$ states, $|1,1,\pm 1\rangle = \mp(a_x^\dagger \pm i a_y^\dagger)|0\rangle/\sqrt{2}$ and $|1,1,0\rangle = a_z^\dagger|0\rangle$, form the $l = 1$ representation. (Notation is $|N,l,m\rangle$.)

   (e) Calculate the expectation values of the quadrupole moment $\langle 1m|(3z^2 - r^2)|1m\rangle$ for $N = 1$, $m = -1, 0, 1$ states, and verify the Wigner–Eckart theorem.

   (f) There are six possible states at $N = 2$ level. Construct states $|2,l,m\rangle$ with definite $l = 0, 2$ and m.

   (g) How many possible states are there at $N = 3, 4$ levels? What l representations do they fall into?

2. Two angular momenta j_1 and j_2 are added to j. Calculate the expectation values of $\langle jm|(\vec{J}_1 \cdot \vec{J}_2)|jm\rangle$. (This is how you calculate the fine splittings in the presence of the spin-orbit interaction in the perturbation theory.)

3. Consider the Stern–Gerlach experiment for spin 1. When the atom enters with $J_z = +\hbar$ in the magnetic field along the y axis, determine the relative strengths of three lines that correspond to $J_y = +\hbar, 0, -\hbar$.