HW #8

1. 3D Harmonic Oscillator

(a)

$\left[L_i, \frac{\vec{p}^2}{2m}\right] = \epsilon_{ijk}[x_j\, p_k, \frac{p_l\, p_l}{2m}] = \epsilon_{ijk}\, i\hbar\, \delta_{jl}(p_k\, p_l + p_l + p_k)\, \frac{1}{2m} = 2\,\epsilon_{ilk}\, p_l\, p_k\, \frac{i\hbar}{2m} = 0,$
$\left[L_i, \frac{1}{2}\, m\,\omega^2\, \vec{x}^2\right] = \epsilon_{ijk}[x_j\, p_k, \frac{1}{2}\, m\,\omega^2\, x_l\, x_l] = \epsilon_{ijk}\, x_j(-i\hbar\,\delta_{kl})\,(x_j\, x_l + x_l\, x_j)\, \frac{1}{2}\, m\,\omega^2 = 2\,\epsilon_{ijl}\, x_j\, x_l\, \frac{-i\hbar}{2}\, m\,\omega^2 = 0.$
In both cases, I used the anti-symmetry of the Levi-Civita symbol ϵ_{ijk}.

(b)

We generalize the usual creation and annihilation operator for each spatial direction, $a_i = \sqrt{\frac{m\omega}{2\hbar}}\,(x_i + i\,\frac{p_i}{m\omega})$, $a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,(x_i - i\,\frac{p_i}{m\omega})$. The commutation relations are obviously $[a_i, a_j^\dagger] = \delta_{ij}$. The Hamiltonian is simply the sum of three harmonic oscillator Hamiltonians, $H = \hbar\,\omega(a_x^\dagger\, a_x + a_y^\dagger\, a_y + a_z^\dagger\, a_z + \frac{3}{2})$.

The angular momentum operators are rewritten using $x_i = \sqrt{\frac{\hbar}{2m\omega}}\,(a_i + a_i^\dagger)$, $p_i = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a_i - a_i^\dagger)$. We find
$L_i = \epsilon_{ijk}\, x_j\, p_k = \epsilon_{ijk}\sqrt{\frac{\hbar}{2m\omega}}\,(a_j + a_j^\dagger)\left(-i\sqrt{\frac{\hbar m\omega}{2}}\right)(a_k - a_k^\dagger)$
$= -i\,\frac{\hbar}{2}\,\epsilon_{ijk}(a_j + a_j^\dagger)\,(a_k - a_k^\dagger) = -i\,\frac{\hbar}{2}\,\epsilon_{ijk}(a_j\, a_k - a_j\, a_k^\dagger + a_j^\dagger\, a_k - a_j^\dagger\, a_k^\dagger)$
$= -i\,\frac{\hbar}{2}\,\epsilon_{ijk}(a_j^\dagger\, a_k - a_k^\dagger\, a_j) = -i\,\hbar\,\epsilon_{ijk}\, a_j^\dagger\, a_k.$
For later purposes, it is useful to define
$a_+ = \frac{-1}{\sqrt{2}}\,(a_x - i\,a_y),\ a_- = \frac{1}{\sqrt{2}}\,(a_x + i\,a_y)$
Note $[a_+, a_+^\dagger] = 1$, $[a_-, a_-^\dagger] = 1$, $[a_+, a_-] = [a_+, a_-^\dagger] = [a_+^\dagger, a_-] = [a_+^\dagger, a_-^\dagger] = 0$. Then the angular momentum operators can be further rewritten as
$L_+ = L_x + i\,L_y = -i\,\hbar(a_y^\dagger\, a_z - a_z^\dagger\, a_y) + \hbar(a_z^\dagger\, a_x - a_x^\dagger\, a_z)$
$= \hbar(-(a_x^\dagger + i\,a_y^\dagger)\,a_z + a_z^\dagger(a_x + i\,a_y)) = \hbar\sqrt{2}\,(a_+^\dagger\, a_z + a_z^\dagger\, a_-)$
$L_- = L_x - i\,L_y = -i\,\hbar(a_y^\dagger\, a_z - a_z^\dagger\, a_y) - \hbar(a_z^\dagger\, a_x - a_x^\dagger\, a_z)$
$= \hbar((a_x^\dagger - i\,a_y^\dagger)\,a_z - a_z^\dagger(a_x - i\,a_y)) = \hbar\sqrt{2}\,(a_-^\dagger\, a_z + a_z^\dagger\, a_+)$
$L_z = -i\,\hbar\,(a_x^\dagger\, a_y - a_y^\dagger\, a_x) = -i\,\hbar\left(\frac{a_-^\dagger - a_+^\dagger}{\sqrt{2}}\,\frac{a_- + a_+}{\sqrt{2}\,i} - \frac{a_-^\dagger + a_+^\dagger}{-\sqrt{2}\,i}\,\frac{a_- - a_+}{\sqrt{2}}\right) = \hbar(a_+^\dagger\, a_+ - a_-^\dagger\, a_-).$
Namely, $a_+^\dagger$ (a_+) creates (annihilates) the excitation with $L_z = +\hbar$, while $a_-^\dagger$ (a_-) creates (annihilates) one with $L_z = -\hbar$.

It is useful to note that the creation operators are spherical tensor operators, $T_{+1}^{(1)} = a_+^\dagger$, $T_0^{(1)} = a_z^\dagger$, $T_{-1}^{(1)} = a_-^\dagger$. To verify this point:
$[L_+, T_{+1}^{(1)}] = \left[\hbar\sqrt{2}\,(a_+^\dagger\, a_z + a_z^\dagger\, a_-), a_+^\dagger\right] = 0,$
$[L_-, T_{+1}^{(1)}] = \left[\hbar\sqrt{2}\,(a_-^\dagger\, a_z + a_z^\dagger\, a_+), a_+^\dagger\right] = \hbar\sqrt{2}\, a_z^\dagger = \hbar\sqrt{2}\, T_0^{(1)}$
$[L_-, T_0^{(1)}] = \left[\hbar\sqrt{2}\,(a_-^\dagger\, a_z + a_z^\dagger\, a_+), a_z^\dagger\right] = \hbar\sqrt{2}\, a_-^\dagger = \hbar\sqrt{2}\, T_{-1}^{(1)},$
$[L_-, T_{-1}^{(1)}] = \left[\hbar\sqrt{2}\,(a_-^\dagger\, a_z + a_z^\dagger\, a_+), a_+^\dagger\right] = 0.$

(c)

The ground state is unique, and the only representation of angular momentum that can be formed by a single state is $l = 0$.

A more explicit way to show it is simply by acting
$L_+ |0\rangle = \hbar\sqrt{2}\,(a_+^\dagger a_z + a_z^\dagger a_-)\,|0\rangle = 0$,
$L_- |0\rangle = \hbar\sqrt{2}\,(a_-^\dagger a_z + a_z^\dagger a_+)\,|0\rangle = 0$,
$L_z |0\rangle = \hbar(a_+^\dagger a_+ - a_-^\dagger a_-)\,|0\rangle = 0$.

(d)

Using the notation defined above, $|1, 1, \pm 1\rangle = a_\pm^\dagger |0\rangle$.
First we show that $|1, 1, 1\rangle$ cannot be raised:
$L_+ |1, 1, 1\rangle = \hbar\sqrt{2}\,(a_+^\dagger a_z + a_z^\dagger a_-)\,a_+^\dagger\,|0\rangle = 0$.
Then by lowering this state,
$L_- |1, 1, 1\rangle = \hbar\sqrt{2}\,(a_-^\dagger a_z + a_z^\dagger a_+)\,a_+^\dagger\,|0\rangle = \hbar\sqrt{2}\,a_z^\dagger\,|0\rangle = \hbar\sqrt{2}\,|1, 1, 0\rangle$.
Lowering this state once more,
$L_- |1, 1, 0\rangle = \hbar\sqrt{2}\,(a_-^\dagger a_z + a_z^\dagger a_+)\,a_z^\dagger\,|0\rangle = \hbar\sqrt{2}\,a_-^\dagger\,|0\rangle = \hbar\sqrt{2}\,|1, 1, -1\rangle$.
Finally this state cannot be lowered
$L_- |1, 1, -1\rangle = \hbar\sqrt{2}\,(a_-^\dagger a_z + a_z^\dagger a_+)\,a_-^\dagger\,|0\rangle = 0$.
Therefore, they form the $l = 1$ representation correctly.

(e)

We first rewrite the quadrupole moment operator in terms of creation and annihilation operators. We start with
$x_i = \sqrt{\frac{\hbar}{2 m \omega}}\,(a_i + a_i^\dagger)$
$3 z^2 - r^2 = 2 z^2 - x^2 - y^2 = \frac{\hbar}{2 m \omega}\left(2\,(a_z + a_z^\dagger)^2 - (a_x + a_x^\dagger)^2 - (a_y + a_y^\dagger)^2\right)$
$= \frac{\hbar}{2 m \omega}\left(2\,(a_z + a_z^\dagger)^2 - \left(\frac{a_- - a_+}{\sqrt{2}} + \frac{a_-^\dagger - a_+^\dagger}{\sqrt{2}}\right)^2 - \left(\frac{a_- + a_+}{\sqrt{2}\,i} + \frac{a_-^\dagger + a_+^\dagger}{-\sqrt{2}\,i}\right)^2\right)$
Because we will take the expectation values of this operator, we are only interested in the pieces with one creation and one annihilation operators. The pieces with two creation or two annihilation operators do not give non-vanishing expectation values. Therefore keeping only those terms,
$3 z^2 - r^2 \approx \frac{\hbar}{2 m \omega}\,(2\,(a_z a_z^\dagger + a_z^\dagger a_z) -$
$\quad \frac{1}{2}\,((a_- - a_+)(a_-^\dagger - a_+^\dagger) + (a_-^\dagger - a_+^\dagger)(a_- - a_+)) - \frac{1}{2}\,((a_- + a_+)(a_-^\dagger + a_+^\dagger) + (a_-^\dagger + a_+^\dagger)(a_- + a_+)))$
$= \frac{\hbar}{2 m \omega}\,(2\,(a_z a_z^\dagger + a_z^\dagger a_z) - (a_- a_-^\dagger + a_-^\dagger a_- + a_+ a_+^\dagger + a_+^\dagger a_+))$
$= \frac{\hbar}{m \omega}\,(2\,a_z^\dagger a_z - (a_-^\dagger a_- + a_+^\dagger a_+))$.
In the last step, we used the commutation relation to rewrite $a a^\dagger = a^\dagger a + 1$, and cancelled the constant pieces against each other. Then it is easy to work out the expectation values,
$\langle 1, 1, 1 | 3 z^2 - r^2 | 1, 1, 1\rangle = \langle 0 | a_+ \frac{\hbar}{m \omega}\,(2\,a_z^\dagger a_z - (a_-^\dagger a_- + a_+^\dagger a_+))\,a_+^\dagger |0\rangle = -\frac{\hbar}{m \omega}$,
$\langle 1, 1, 0 | 3 z^2 - r^2 | 1, 1, 0\rangle = \langle 0 | a_z \frac{\hbar}{m \omega}\,(2\,a_z^\dagger a_z - (a_-^\dagger a_- + a_+^\dagger a_+))\,a_z^\dagger |0\rangle = 2\,\frac{\hbar}{m \omega}$,
$\langle 1, 1, -1 | 3 z^2 - r^2 | 1, 1, -1\rangle = \langle 0 | a_- \frac{\hbar}{m \omega}\,(2\,a_z^\dagger a_z - (a_-^\dagger a_- + a_+^\dagger a_+))\,a_-^\dagger |0\rangle = -\frac{\hbar}{m \omega}$.
The quadrupole moment operator here is a spherical tensor operator with $k = 2$, $q = 0$. To see if this result is consistent with the Wigner-Eckart theorem, we need the Clebsch-Gordan coefficients

```
Table[ClebschGordan[{1, m}, {2, 0}, {1, m}], {m, -1, 1}]
```

$$\left\{\frac{1}{\sqrt{10}}, -\sqrt{\frac{2}{5}}, \frac{1}{\sqrt{10}}\right\}$$

The ratios among the expectation values are indeed the same as the ratios among the Clebsch-Gordan coefficients, $1 : -2 : 1$.

As an added note, a positive quadrupole moment $\langle 3 z^2 - r^2 \rangle > 0$ indicates a prolate form, while a negative quadrupole moment $\langle 3 z^2 - r^2 \rangle < 0$ indicates an oblated form. This is consistent with the pictures obtained in HW#7.

(f)

The six states are: $(a_+^\dagger)^2 |0\rangle$, $(a_z^\dagger)^2 |0\rangle$, $(a_-^\dagger)^2 |0\rangle$, $a_+^\dagger a_z^\dagger |0\rangle$, $a_+^\dagger a_-^\dagger |0\rangle$, $a_z^\dagger a_-^\dagger |0\rangle$. By looking at the L_z eigenvalues, it is easy to identify
$|2, 2, 2\rangle = \frac{1}{\sqrt{2}} (a_+^\dagger)^2 |0\rangle$,
$|2, 2, 1\rangle = a_+^\dagger a_z^\dagger |0\rangle$,
$|2, 2, -1\rangle = a_z^\dagger a_-^\dagger |0\rangle$,
$|2, 2, -2\rangle = \frac{1}{\sqrt{2}} (a_-^\dagger)^2 |0\rangle$.
There are two states with $m = 0$: $(a_z^\dagger)^2 |0\rangle$, $a_+^\dagger a_-^\dagger |0\rangle$. We can tell which linear combination belongs to $l = 2$ representation by acting L_- on $|2, 2, 1\rangle$,
$L_- |2, 2, 1\rangle = \hbar \sqrt{2} (a_-^\dagger a_z + a_z^\dagger a_+) a_+^\dagger a_z^\dagger |0\rangle = \hbar \sqrt{2} (a_-^\dagger a_+^\dagger + a_z^\dagger a_z^\dagger) |0\rangle = \hbar \sqrt{6} |2, 2, 0\rangle$.
Therefore, we can identify
$|2, 2, 0\rangle = \frac{1}{\sqrt{3}} (a_-^\dagger a_+^\dagger + a_z^\dagger a_z^\dagger) |0\rangle$
which is properly normalized as it should be. The orthogonal combination is
$|2, 0, 0\rangle = \frac{1}{\sqrt{6}} (2 a_-^\dagger a_+^\dagger - a_z^\dagger a_z^\dagger) |0\rangle$.
To verify that this state is indeed an $l = 2$ state, we can check
$L_+ |2, 0, 0\rangle = \hbar \sqrt{2} (a_+^\dagger a_z + a_z^\dagger a_-) \frac{1}{\sqrt{6}} (2 a_-^\dagger a_+^\dagger - a_z^\dagger a_z^\dagger) |0\rangle = \hbar \sqrt{2} \frac{1}{\sqrt{6}} (-a_+^\dagger a_z^\dagger - a_z^\dagger a_+^\dagger + 2 a_z^\dagger a_+^\dagger) |0\rangle = 0$

A much more systematic way of obtaining the same result is to use Sakurai's Eq. (3.10.27). Even though this example is simple enough to work it out explicitly as I did above, the generalization to higher N would be quite cumbersome.

Eq. (3.10.27) says
$T_0^{(2)} = \sum_{q_1, q_2} \langle 1\, 1; q_1, q_2 | 2\, 0\rangle T_{q_1}^{(1)} T_{q_2}^{(1)}$
$= \langle 1\, 1; +1\ -1 | 2\, 0\rangle T_{+1}^{(1)} T_{-1}^{(1)} + \langle 1\, 1; 0\, 0 | 2\, 0\rangle T_0^{(1)} T_0^{(1)} + \langle 1\, 1; -1\ +1 | 2\, 0\rangle T_{-1}^{(1)} T_{+1}^{(1)}$
$= \frac{1}{\sqrt{6}} a_+^\dagger a_-^\dagger + \sqrt{\frac{2}{3}} a_z^\dagger a_z^\dagger + \frac{1}{\sqrt{6}} a_-^\dagger a_+^\dagger$
$= \sqrt{\frac{2}{3}} (a_-^\dagger a_+^\dagger + a_z^\dagger a_z^\dagger)$.
Therefore, the operator $(a_-^\dagger a_+^\dagger + a_z^\dagger a_z^\dagger)$ creates an $l = 2$ state. Similarly,
$T_0^{(0)} = \sum_{q_1, q_2} \langle 1\, 1; q_1, q_2 | 0\, 0\rangle T_{q_1}^{(1)} T_{q_2}^{(1)}$
$= \langle 1\, 1; +1\ -1 | 0\, 0\rangle T_{+1}^{(1)} T_{-1}^{(1)} + \langle 1\, 1; 0\, 0 | 0\, 0\rangle T_0^{(1)} T_0^{(1)} + \langle 1\, 1; -1\ +1 | 0\, 0\rangle T_{-1}^{(1)} T_{+1}^{(1)}$
$= \frac{1}{\sqrt{3}} a_+^\dagger a_-^\dagger - \frac{1}{\sqrt{3}} a_z^\dagger a_z^\dagger + \frac{1}{\sqrt{3}} a_-^\dagger a_+^\dagger$
$= \frac{1}{\sqrt{3}} (2 a_-^\dagger a_+^\dagger - a_z^\dagger a_z^\dagger) = -\frac{1}{\sqrt{3}} (a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger + a_z^\dagger a_z^\dagger) = -\frac{1}{\sqrt{3}} \vec{a}^\dagger \cdot \vec{a}^\dagger$.
The last expression shows it is manifestly rotation invariant. Therefore, the operator $(2 a_-^\dagger a_+^\dagger - a_z^\dagger a_z^\dagger)$ creates an $l = 0$ state. The rest of the job is to properly normalize the states, reproducing the above results.

(g)

For $N = 3$. The number of states is given by the number of combinations to choose three out of three, allowing for multiple picks. Using the general formula ${}_n H_r = {}_{n+r-1} C_r$, ${}_3 H_3 = {}_5 C_3 = 10$. It is clear that the state with the highest L_z eigenvalue $(a_+{}^\dagger)^3 |0\rangle$ has $m = 3$ and hence belongs to the $l = 3$ representation, and it has $2\,l + 1 = 7$ states. The remaining $10 - 7 = 3$ states then must form the $l = 1$ representation.

For $N = 4$. The number of states is given by the number of combinations to choose three out of four, allowing for multiple picks. Using the general formula ${}_n H_r = {}_{n+r-1} C_r$, ${}_3 H_4 = {}_6 C_4 = 15$. It is clear that the state with the highest L_z eigenvalue $(a_+{}^\dagger)^4 |0\rangle$ has $m = 4$ and hence belongs to the $l = 4$ representation, and it has $2\,l + 1 = 9$ states. The remaining $15 - 9 = 6$ states then must form the $l = 2$ and $l = 0$ representations.

Note that the creation operators are linear combinations of $\vec{x}$ and $\vec{p}$ and hence parity odd. Therefore, $N =$ even states have even parity, and hence can only have even l, while $N =$ odd states odd parity, and hence odd l. In general, $N =$ even states have $l = 0, 2, \cdots, N$, while $N =$ odd states have $l = 1, 3, \cdots, N$. It can be verified by looking at the number of states. The number of states at level N is ${}_3 H_N = {}_{N+2} C_N = (N + 2)(N + 1)/2$. For even $N = 2\,k$, it is $(k + 1)(2\,k + 1)$. Each $l = 2\,n$ contributes $2\,l + 1 = 4\,n + 1$ states, and the total is $\sum_{n=0}^{k} (4\,n + 1) = 2\,k(k + 1) + (k + 1) = (k + 1)(2\,k + 1)$. For odd $N = 2\,k - 1$, the number of states is $k(2\,k + 1)$. Each $l = 2\,n - 1$ contributes $2\,l + 1 = 4\,n - 1$ states, and the total is $\sum_{n=1}^{k} (4\,n - 1) = 2\,k(k + 1) - k = k(2\,k + 1)$.

2. Inner product of angular momentum operators

Note that $\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}\left(\left(\vec{J}_1 + \vec{J}_2\right)^2 - \vec{J}_1{}^2 - \vec{J}_2{}^2\right)$. In our case, we know that the states of our interest have eigenvalues $\vec{J}_1{}^2 = \hbar^2\, j_1(j_1 + 1)$, $\vec{J}_2{}^2 = \hbar^2\, j_2(j_2 + 1)$. The total angular momentum is j, and hence $\left(\vec{J}_1 + \vec{J}_2\right)^2 = \vec{J}^2 = \hbar^2\, j(j + 1)$. Therefore, $\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}\hbar^2(j(j + 1) - j_1(j_1 + 1) - j_2(j_2 + 1))$.

It is instructive to verify that $\mathrm{Tr}\left(\vec{J}_1 \cdot \vec{J}_2\right) = \mathrm{Tr}\,\vec{J}_1 \cdot \mathrm{Tr}\,\vec{J}_2 = 0$. In this definition of the trace on the left-hand side, it needs to include the entire Hilbert space $j = |j_1 - j_2|, |j_1 - j_2| + 1, \cdots, j_1 + j_2$. For the sake of definiteness, we can always choose $j_1 > j_2$ without a loss of generality. Then, $\mathrm{Tr}\left(\vec{J}_1 \cdot \vec{J}_2\right) = \sum_{j=j_1-j_2}^{j_1+j_2} (2\,j + 1)\,\frac{1}{2}\,\hbar^2(j(j + 1) - j_1(j_1 + 1) - j_2(j_2 + 1))$.

```
Sum[(2 j + 1) j (j + 1), {j, j1 - j2, j1 + j2}]
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$(1 + 2\,j_1)\,(1 + 2\,j_2)\,(j_1 + j_1^2 + j_2 + j_2^2)$

```
Sum[(2 j + 1), {j, j1 - j2, j1 + j2}]
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$1 + 2\,j_2 + 2\,j_1\,(1 + 2\,j_2)$

```
Simplify[Expand[%% - % (j1 (j1 + 1) + j2 (j2 + 1))]]
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0

Therefore, $\mathrm{Tr}\left(\vec{J}_1 \cdot \vec{J}_2\right) = 0$ as expected.

3. Stern-Gerlach Experiment

One way is to find the eigenstates of J_y in the J_z-representation. Starting with the expression we found in HW#7,

$$J_+ = \hbar\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{2}\\ 0 & 0 & 0\end{pmatrix},\ J_- = \hbar\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{2} & 0\end{pmatrix},$$

we find

$$J_y = \frac{J_+ - J_-}{2i} = \hbar\,\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}.$$

The eigenstates can be obtained by

```
Eigensystem[1/√2 {{0, -I, 0}, {I, 0, -I}, {0, I, 0}}]

{{-1, 0, 1}, {{-1, ⅈ √2, 1}, {1, 0, 1}, {-1, -ⅈ √2, 1}}}
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The properly normalized J_y eigenstates are therefore

$$|J_y = +\hbar\rangle = \frac{1}{2}\begin{pmatrix}-1\\ i\sqrt{2}\\ 1\end{pmatrix},\ |J_y = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\ |J_y = -\hbar\rangle = \frac{1}{2}\begin{pmatrix}-1\\ -i\sqrt{2}\\ 1\end{pmatrix}.$$

The initial state

$$|J_z = +\hbar\rangle = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

can then be expanded as

$$\begin{aligned}|J_z = +\hbar\rangle &= |J_y = +\hbar\rangle\langle J_y = +\hbar\,|\,J_z = +\hbar\rangle + |J_y = 0\rangle\langle J_y = 0\,|\,J_z = +\hbar\rangle + |J_y = -\hbar\rangle\langle J_y = -\hbar\,|\,J_z = +\hbar\rangle\\ &= -\frac{1}{2}|J_y = +\hbar\rangle + \frac{1}{\sqrt{2}}|J_y = 0\rangle - \frac{1}{2}|J_y = -\hbar\rangle\end{aligned}$$

The probabilities to find $J_y = +\hbar,\ 0,\ -\hbar$ are therefore $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively, and hence the relative strengths of three lines are 1:2:1.

Another way to obtain the same result is to use the rotation matrices in Sakurai (3.5.57). To rotate the J_z eigenstates to J_y eigenstates, we need to rotate the system around the x-axis by $\pi/2$. Using the Euler rotations, it is achieved by $\pi/2$ around the z-axis, $\pi/2$ rotation around the y-axis, and rotating back by $\pi/2$ around the z-axis. Therefore the matrix is

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In[48]:= {{1/2 (1 + Cos[β]), -1/√2 Sin[β], 1/2 (1 - Cos[β])}, {1/√2 Sin[β], Cos[β], -1/√2 Sin[β]},
           {1/2 (1 - Cos[β]), 1/√2 Sin[β], 1/2 (1 + Cos[β])}} /. {β → -π/2}

Out[48]= {{1/2, 1/√2, 1/2}, {-1/√2, 0, 1/√2}, {1/2, -1/√2, 1/2}}

In[49]:= DiagonalMatrix[{E^(-I π/2), 1, E^(I π/2)}].%.DiagonalMatrix[{E^(I π/2), 1, E^(-I π/2)}]

Out[49]= {{1/2, -ⅈ/√2, -1/2}, {-ⅈ/√2, 0, -ⅈ/√2}, {-1/2, -ⅈ/√2, 1/2}}
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In[50]:= % // MatrixForm
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Out[50]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{i}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

Up to overall phase factors, the three column vectors $\begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ 0 \\ -\frac{i}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$ precisely agree with the eigenstates of J_y obtained above.