HW #5 (221A), due Oct 1, 4pm

Answer the following questions on the harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

(a) Verify that the Hamiltonian can be recast to the form $H = \hbar\omega(N + \frac{1}{2})$, where $N = a^\dagger a$ and

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right). \tag{2}$$

(b) Write down the condition for the ground state $a|0\rangle = 0$ in the position representation, solve it analytically, and normalize it properly. Plot the shape of the wave function.

(c) Work out the wave functions of the first- and second-excited states $\langle x|1\rangle$ and $\langle x|2\rangle$ using the ground state wave function and the creation operator. Pay attention to the normalization. Plot the shapes of the wave functions.

(d) Without using the explicit forms of the wave function, show that $\langle x\rangle = 0$ and $(\Delta x)^2 = \frac{\hbar}{2m\omega}(2n+1)$ for the state $|n\rangle$.

(e) Similarly, work out $\langle p\rangle$ and $(\Delta p)^2$ for the state $|n\rangle$ and the product $(\Delta x)(\Delta p)$.

(f) Verify that the coherent state

$$|f\rangle = e^{fa^\dagger}|0\rangle \tag{3}$$

is an eigenstate of the annihilation operator $a|f\rangle = f|f\rangle$. Note that f is in general a *complex* number.

(g) Calculate $\langle x\rangle$, $\langle p\rangle$, $(\Delta x)^2$, and $(\Delta p)^2$ for the coherent state.

(h) Work out the time evolution of the coherent state $|f, t\rangle$ (in the Schrödinger picture) and the expectation value $\langle f, t|x|f, t\rangle$.

(i) Solve the Heisenberg equation of motion, and use it to calculate $\langle f|x(t)|f\rangle$ in the Heisenberg picture.