HW #3 (221A), due Sep 17, 4pm

1. Answer the following questions about the spin operators using their matrix representation.

(a) Show the commutation relations \([S_i, S_j] = i\hbar \epsilon_{ijk} S_k\).

(b) Construct the eigenstates of the spin operator along an arbitrary axis \(\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\), and calculate the probability that the state with spin along the \(\vec{n}\) direction would be measured to have the positive spin along the \(z\)-axis.

(c) Generalize the result to the spin along the \(\vec{n}\) direction measured along the \(\vec{n}'\) direction, and show that the result depends only on the angle between \(\vec{n}\) and \(\vec{n}'\) as expected from the rotational invariance.

2. Here is a sloppy way to estimate the energy of the hydrogen atom using the uncertainty principle. Using the energy

\[ E = \frac{p^2}{2m} - \frac{Ze^2}{r}, \quad (1) \]

and assuming that the “size” (\(\Delta x\)) of the wave function is approximately \(d\), the typical order of magnitude of the momentum is given by \(p = \Delta p \simeq \hbar/d\). (Here I assumed that the momentum distribution is peaked at zero symmetrically, so that I can equate \(p\) and \(\Delta p\)). The typical size of the potential energy is then \(Ze^2/d\). Minimize the energy with respect to \(d\) and find the “ground state energy.” Compare the estimate to the exact result.

3. The uncertainty relation is actually there in classical waves, too.

(a) Show that a “localized light” of the form

\[ E_y(x, t) = E_0 \sin 2\pi \nu \left(t - \frac{x}{c}\right) e^{-\frac{(x-ct)^2}{2\sigma^2}} \quad (2) \]

is a solution to the (one-dimensional) Maxwell equation. Sketch its shape. Calculate the “uncertainty” in the position \(\Delta x\). 

(b) Using Fourier analysis, determine the frequency of the light and its dispersion \(\Delta \nu\). What is the product \(\Delta x \Delta \nu\)?