HW #3 (221A), due Sep 17, 4pm

1. Answer the following questions about the spin operators using their matrix representation.

   (a) Show the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$.

   (b) Construct the eigenstates of the spin operator along an arbitrary axis $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and calculate the probability that the state with spin along the $\vec{n}$ direction would be measured to have the positive spin along the z-axis.

   (c) Generalize the result to the spin along the $\vec{n}$ direction measured along the $\vec{n}'$ direction, and show that the result depends only on the angle between $\vec{n}$ and $\vec{n}'$ as expected from the rotational invariance.

2. Here is a sloppy way to estimate the energy of the hydrogen atom using the uncertainty principle. Using the energy

$$E = \frac{\vec{p}^2}{2m} - \frac{Ze^2}{r}, \tag{1}$$

   and assuming that the "size" (Δx) of the wave function is approximately d, the typical order of magnitude of the momentum is given by $p = \Delta p \simeq \hbar/d$. (Here I assumed that the momentum distribution is peaked at zero symmetrically, so that I can equate p and Δp). The typical size of the potential energy is then Ze^2/d. Minimize the energy with respect to d and find the "ground state energy." Compare the estimate to the exact result.

3. The uncertainty relation is actually there in classical waves, too.

   (a) Show that a "localized light" of the form

$$E_y(x,t) = E_0 \sin 2\pi\nu \left(t - \frac{x}{c}\right) e^{-(x-ct)^2/2\sigma^2} \tag{2}$$

   is a solution to the (one-dimensional) Maxwell equation. Sketch its shape. Calculate the "uncertainty" in the position Δx.

   (b) Using Fourier analysis, determine the frequency of the light and its dispersion $\Delta\nu$. What is the product $\Delta x \Delta\nu$?