HW #3

1. Spin Matrices

We use the spin operators represented in the bases where S_z is diagonal:

```
Sx = ħ/2 {{0, 1}, {1, 0}}; Sy = ħ/2 {{0, -I}, {I, 0}}; Sz = ħ/2 {{1, 0}, {0, -1}};
```

```
Sx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

```
Sy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}$$

```
Sz // MatrixForm
```

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

(a) Obviously two matrices commute when they are the same: $i = j$. Also, it is obvious that $[S_i, S_j]$ is anti-symmetric in $i \longleftrightarrow j$ because $[S_j, S_i] = -[S_i, S_j]$. Therefore, it only reamains to verify

```
Sx.Sy - Sy.Sx - I ħ Sz
```

```
{{0, 0}, {0, 0}}
```

```
Sy.Sz - Sz.Sy - I ħ Sx
```

```
{{0, 0}, {0, 0}}
```

```
Sz.Sx - Sx.Sz - I ħ Sy
```

```
{{0, 0}, {0, 0}}
```

(b) We define $\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$

```
nx = Sin[θ] Cos[ϕ]; ny = Sin[θ] Sin[ϕ]; nz = Cos[θ]
```

```
Cos[θ]
```

```
Sn = Simplify[nx Sx + ny Sy + nz Sz]
```

$$\left\{\left\{\frac{1}{2}\hbar\,\mathrm{Cos}[\theta],\ \frac{1}{2}\hbar\,\mathrm{Sin}[\theta]\,(\mathrm{Cos}[\phi] - i\,\mathrm{Sin}[\phi])\right\},\ \left\{\frac{1}{2}\hbar\,\mathrm{Sin}[\theta]\,(\mathrm{Cos}[\phi] + i\,\mathrm{Sin}[\phi]),\ -\frac{1}{2}\hbar\,\mathrm{Cos}[\theta]\right\}\right\}$$

```
Eigensystem[S_n]
```

$$\left\{\left\{-\frac{\sqrt{\hbar^2}}{2}, \frac{\sqrt{\hbar^2}}{2}\right\}, \left\{\left\{\frac{\left(-\sqrt{\hbar^2}+\hbar\,\mathrm{Cos}[\theta]\right)\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}, \left\{\frac{\left(\sqrt{\hbar^2}+\hbar\,\mathrm{Cos}[\theta]\right)\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}\right\}\right\}$$

```
PowerExpand[%]
```

$$\left\{\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\}, \left\{\left\{\frac{(-\hbar+\hbar\,\mathrm{Cos}[\theta])\,\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}, \left\{\frac{(\hbar+\hbar\,\mathrm{Cos}[\theta])\,\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}\right\}\right\}$$

```
Simplify[%]
```

$$\left\{\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\}, \left\{\left\{(-\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])\,\mathrm{Tan}\left[\frac{\theta}{2}\right], 1\right\}, \left\{\mathrm{Cot}\left[\frac{\theta}{2}\right](\mathrm{Cos}[\phi]-\mathrm{i}\,\mathrm{Sin}[\phi]), 1\right\}\right\}\right\}$$

Therefore, one can take the the normalized eigenstates to be $\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$ with eigenvalue $+\frac{\hbar}{2}$ and $\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$ with eigenvalue $-\frac{\hbar}{2}$. The state with spin along the $\vec{n}$ direction is the former, and its probability to have the positive S_z when measured is simply given by $|\langle S_z=+\frac{\hbar}{2}\,|\,S_n=+\frac{\hbar}{2}\rangle|^2 = \left|(1,0)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}\right|^2 = \cos^2\frac{\theta}{2}$.

(c) Between $\vec{n}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\vec{n}'=(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$, the probability is $|\langle S_{n'}=+\frac{\hbar}{2}\,|\,S_n=+\frac{\hbar}{2}\rangle|^2 = \left|(\cos\frac{\theta'}{2}, \sin\frac{\theta'}{2}e^{-i\phi'})\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}\right|^2 = |\cos\frac{\theta'}{2}\cos\frac{\theta}{2}+\sin\frac{\theta'}{2}e^{-i\phi'}\sin\frac{\theta}{2}e^{i\phi}|^2 =$

$$\cos^2\frac{\theta'}{2}\cos^2\frac{\theta}{2}+\sin^2\frac{\theta'}{2}\sin^2\frac{\theta}{2}+2\cos\frac{\theta'}{2}\cos\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\cos(\phi-\phi')$$

TrigExpand$\left[\mathrm{Cos}\left[\frac{\theta_1}{2}\right]^2\mathrm{Cos}\left[\frac{\theta_2}{2}\right]^2+\mathrm{Sin}\left[\frac{\theta_1}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_2}{2}\right]^2+2\,\mathrm{Cos}\left[\frac{\theta_1}{2}\right]\mathrm{Cos}\left[\frac{\theta_2}{2}\right]\mathrm{Sin}\left[\frac{\theta_1}{2}\right]\mathrm{Sin}\left[\frac{\theta_2}{2}\right]\mathrm{Cos}[\phi_1-\phi_2]\right]$

$$\frac{1}{2}+\frac{1}{2}\mathrm{Cos}\left[\frac{\theta_1}{2}\right]^2\mathrm{Cos}\left[\frac{\theta_2}{2}\right]^2-\frac{1}{2}\mathrm{Cos}\left[\frac{\theta_2}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_1}{2}\right]^2+$$
$$2\,\mathrm{Cos}\left[\frac{\theta_1}{2}\right]\mathrm{Cos}\left[\frac{\theta_2}{2}\right]\mathrm{Cos}[\phi_1]\,\mathrm{Cos}[\phi_2]\,\mathrm{Sin}\left[\frac{\theta_1}{2}\right]\mathrm{Sin}\left[\frac{\theta_2}{2}\right]-\frac{1}{2}\mathrm{Cos}\left[\frac{\theta_1}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_2}{2}\right]^2+$$
$$\frac{1}{2}\mathrm{Sin}\left[\frac{\theta_1}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_2}{2}\right]^2+2\,\mathrm{Cos}\left[\frac{\theta_1}{2}\right]\mathrm{Cos}\left[\frac{\theta_2}{2}\right]\mathrm{Sin}\left[\frac{\theta_1}{2}\right]\mathrm{Sin}\left[\frac{\theta_2}{2}\right]\mathrm{Sin}[\phi_1]\,\mathrm{Sin}[\phi_2]$$

```
Simplify[%]
```

$$\frac{1}{2}\left(1+\mathrm{Cos}[\theta_1]\,\mathrm{Cos}[\theta_2]+\mathrm{Cos}[\phi_1]\,\mathrm{Cos}[\phi_2]\,\mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2]+\mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2]\,\mathrm{Sin}[\phi_1]\,\mathrm{Sin}[\phi_2]\right)$$

This is nothing but $\frac{1}{2}\left(1+\vec{n}\cdot\vec{n}'\right)=\frac{1}{2}(1+\cos\eta)=\cos^2\frac{\eta}{2}$, where η is the angle between two vectors, as expected from the rotational invariance.

2. Sloppy Hydrogen Atom

According to the problem,

```
Energy = 1/(2 m) (ħ/d)^2 - Z e^2/d
```

$$-\frac{e^2 Z}{d}+\frac{\hbar^2}{2 d^2 m}$$

```
Solve[D[Energy, d] == 0, d]
```

$$\left\{\left\{d\to \frac{\hbar^2}{e^2 m Z}\right\}\right\}$$

```
Simplify[Energy /. %[[1]]]
```

$$-\frac{e^4 m Z^2}{2 \hbar^2}$$

This actually agrees with the exact result. (One should be cautioned, however, that the agreement with the exact result is a coincidence for this particular example.)

3. Classical Uncertainty Principle

(a) The Maxwell's equations in vacuum are given by

$$\vec{\nabla}\cdot\vec{E}=0$$

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}+\partial_t\vec{B}=0$$

$$c^2\,\vec{\nabla}\times\vec{B}-\partial_t\vec{E}=0$$

In this problem, there are only x and t dependence, and the only non-vanishing components are E_y and B_z. Then the Maxwell's equations reduce to

$$\nabla_x E_y+\partial_t B_z=0$$

$$-c^2\,\nabla_x B_z-\partial_t E_y=0$$

Putting them together, they reduce to a simple one-dimensional equation,

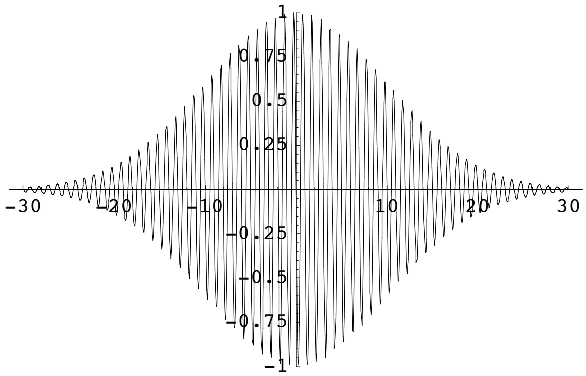
$$c^2\,\nabla_x{}^2 E_y-\partial_t{}^2 E_y=0.$$

Any function of the combination $c\,t-x$ satisfies this equation, namely

$$(c^2\,\nabla_x{}^2-\partial_t{}^2)\,f(c\,t-x)=0.$$

Because the form of E_y given in the problem is a function of $c\,t-x$ only, it solves the Maxwell's equations automatically. The form can be sketched as

```
Plot[Sin[2 π ν (t - x/c)] E^(-(x - c t)^2/2/σ^2) /. {t → 0, ν → 1, c → 1, σ → 10},
  {x, -30, 30}, PlotRange → {-1, 1}]
```



- Graphics -

It oscillates just like the plane waves, but is localized. The "uncertainty" is defined using the formula analogous to the quantum mechanical wave function. First the "norm,"

```
Integrate[(Sin[2 π ν (t - x/c)] E^(-(x - c t)^2/2/σ^2))^2 /. {t → 0}, {x, -∞, ∞}]
```

$$\text{If}\left[\text{Im}\left[\frac{\nu}{c}\right] == 0 \,\&\&\, \text{Arg}[\sigma] > -\frac{\pi}{4} \,\&\&\, \text{Re}[\sigma^2] > 0 \,\&\&\, \text{Arg}[\sigma] < \frac{\pi}{4},\ \frac{1}{2}\left(1 - e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)\sqrt{\pi}\,\sqrt{\sigma^2},\ \int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}} \,\text{Sin}\left[\frac{2\pi x \nu}{c}\right]^2 dx\right]$$

```
norm = Simplify[PowerExpand[1/2 (1 - e^(-4 π^2 ν^2 σ^2/c^2)) Sqrt[π] Sqrt[σ^2]]]
```

$$\frac{1}{2}\left(1 - e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)\sqrt{\pi}\,\sigma$$

We set the overall normalization $E_0 = 1$ throughout as it drops out after taking the norm correctly into account.

Next the expectation value

```
Integrate[(Sin[2 π ν (t - x/c)] E^(-(x - c t)^2/2/σ^2))^2 x /. {t → 0}, {x, -∞, ∞}]
```

$$\text{If}\left[\text{Re}[\sigma^2] > 0,\ 0,\ \int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}}\, x\, \text{Sin}\left[\frac{2\pi x \nu}{c}\right]^2 dx\right]$$

OK, this vanishes. Finally the variance,

```
Integrate[(Sin[2 π ν (t - x/c)] E^(-(x - c t)^2/2/σ^2))^2 x^2 /. {t → 0}, {x, -∞, ∞}]
```

$$\text{If}\left[\text{Im}\left[\frac{\nu}{c}\right] == 0 \,\&\&\, \text{Arg}[\sigma] > -\frac{\pi}{4} \,\&\&\, \text{Re}[\sigma^2] > 0 \,\&\&\, \text{Arg}[\sigma] < \frac{\pi}{4},\ \frac{e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\sqrt{\pi}\,(\sigma^2)^{3/2}\left(c^2\left(-1 + e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right) + 8\pi^2\nu^2\sigma^2\right)}{4\,c^2},\ \int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}}\, x^2\, \text{Sin}\left[\frac{2\pi x \nu}{c}\right]^2 dx\right]$$

```
Simplify[PowerExpand[ e^(-4π²ν²σ²/c²) √π (σ²)^(3/2) (c² (-1 + e^(4π²ν²σ²/c²)) + 8π²ν²σ²) / (4 c²) / norm]]
```

$$\frac{\sigma^2 \left(c^2 \left(-1+e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)+8\pi^2\nu^2\sigma^2\right)}{2c^2\left(-1+e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)}$$

One can write it as $(\Delta x)^2 = \frac{1}{2}\sigma^2(1-e^{-\gamma}) + \frac{2\gamma e^{-\gamma}}{1-e^{-\gamma}}$, where $\gamma = 4\pi^2\nu^2\sigma^2/c^2$. It is especially simple when $\gamma \gg 1$, when $(\Delta x)^2 = \frac{1}{2}\sigma^2$.

(b) The Fourier transform to the frequency domain is given by

```
Integrate[Sin[2 π ν (t - x/c)] E^(-(x - c t)^2/2/σ^2) E^(I 2 π f t) /. {x → 0},
 {t, -∞, ∞}, Assumptions → {Re[c²/σ²] > 0}]
```

$$\text{If}\left[\text{Im}[f-\nu]==0\ \&\&\ \text{Im}[f+\nu]==0,\ -\frac{i\, e^{-\frac{4\pi^2(f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2(f-\nu)^2\sigma^2}{c^2}}-e^{\frac{2\pi^2(f+\nu)^2\sigma^2}{c^2}}\right)\sqrt{\frac{\pi}{2}}}{\sqrt{\frac{c^2}{\sigma^2}}},\ \int_{-\infty}^{\infty} e^{2 i f \pi t-\frac{c^2t^2}{2\sigma^2}}\,\text{Sin}[2\pi t\nu]\,dt\right]$$

```
Simplify[PowerExpand[- i e^(-4π²(f²+ν²)σ²/c²) (e^(2π²(f-ν)²σ²/c²) - e^(2π²(f+ν)²σ²/c²)) √(π/2) / √(c²/σ²)]]
```

$$-\frac{i\, e^{-\frac{4\pi^2(f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2(f-\nu)^2\sigma^2}{c^2}}-e^{\frac{2\pi^2(f+\nu)^2\sigma^2}{c^2}}\right)\sqrt{\frac{\pi}{2}}\,\sigma}{c}$$

Again starting with the norm(dropping the overall f-independent factors),

```
Integrate[(e^(-4π²(f²+ν²)σ²/c²) (e^(2π²(f-ν)²σ²/c²) - e^(2π²(f+ν)²σ²/c²)))², {f, 0, ∞}]
```

$$\text{If}\left[\text{Re}\left[\frac{\sigma^2}{c^2}\right]>0,\ \frac{e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\left(-1+e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)}{2\sqrt{\pi}\sqrt{\frac{\sigma^2}{c^2}}},\ \int_0^{\infty} e^{-\frac{8\pi^2(f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2(f-\nu)^2\sigma^2}{c^2}}-e^{\frac{2\pi^2(f+\nu)^2\sigma^2}{c^2}}\right)^2 df\right]$$

```
norm = Simplify[PowerExpand[e^(-4π²ν²σ²/c²) (-1 + e^(4π²ν²σ²/c²)) / (2 √π √(σ²/c²))]]
```

$$\frac{c\left(1-e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)}{2\sqrt{\pi}\,\sigma}$$

Next, the average frequency,

$$\textbf{Integrate}\left[\left(e^{-\frac{4\pi^2 (f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2 (f-\nu)^2\sigma^2}{c^2}} - e^{\frac{2\pi^2 (f+\nu)^2\sigma^2}{c^2}}\right)\right)^2 f,\ \{f,\ 0,\ \infty\}\right]$$

$$\text{If}\left[\text{Re}\left[\frac{\sigma^2}{c^2}\right] > 0,\ \frac{\nu^2\,\text{Erf}\left[2\pi\sqrt{\frac{\nu^2\sigma^2}{c^2}}\right]}{2\sqrt{\pi}\sqrt{\frac{\nu^2\sigma^2}{c^2}}},\ \int_0^\infty e^{-\frac{8\pi^2(f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2(f-\nu)^2\sigma^2}{c^2}} - e^{\frac{2\pi^2(f+\nu)^2\sigma^2}{c^2}}\right)^2 f\,df\right]$$

$$\textbf{Simplify}\left[\textbf{PowerExpand}\left[\frac{\nu^2\,\text{Erf}\left[2\pi\sqrt{\frac{\nu^2\sigma^2}{c^2}}\right]}{2\sqrt{\pi}\sqrt{\frac{\nu^2\sigma^2}{c^2}}}\right]\Big/\textbf{norm}\right]$$

$$\frac{\nu\,\text{Erf}\left[\frac{2\pi\nu\sigma}{c}\right]}{1 - e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}}$$

One can write it as $\nu\,\frac{\text{Erf}[\gamma^{1/2}]}{1-E^{-\gamma}}$, where $\gamma = 4\pi^2\,\nu^2\,\sigma^2/c^2$. It is especially simple when $\gamma \gg 1$, when it reduces to nothing but ν. Finally the dispersion in the frequency is

$$\textbf{Integrate}\left[\left(e^{-\frac{4\pi^2 (f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2 (f-\nu)^2\sigma^2}{c^2}} - e^{\frac{2\pi^2 (f+\nu)^2\sigma^2}{c^2}}\right)\right)^2 f^2,\ \{f,\ 0,\ \infty\}\right]$$

$$\text{If}\left[\text{Re}\left[\frac{\sigma^2}{c^2}\right] > 0,\ \frac{e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\left(-c^2 + c^2 e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}} + 8\,e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\pi^2\nu^2\sigma^2\right)}{16\,\pi^{5/2}\,\sigma^2\sqrt{\frac{\sigma^2}{c^2}}},\ \int_0^\infty e^{-\frac{8\pi^2(f^2+\nu^2)\sigma^2}{c^2}}\left(e^{\frac{2\pi^2(f-\nu)^2\sigma^2}{c^2}} - e^{\frac{2\pi^2(f+\nu)^2\sigma^2}{c^2}}\right)^2 f^2\,df\right]$$

$$\textbf{Simplify}\left[\textbf{PowerExpand}\left[\frac{e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}\left(-c^2 + c^2 e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}} + 8\,e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\pi^2\nu^2\sigma^2\right)}{16\,\pi^{5/2}\,\sigma^2\sqrt{\frac{\sigma^2}{c^2}}}\Big/\textbf{norm} - \left(\frac{\nu\,\text{Erf}\left[\frac{2\pi\nu\sigma}{c}\right]}{1 - e^{-\frac{4\pi^2\nu^2\sigma^2}{c^2}}}\right)^2\right]\right]$$

$$\left(\left(-1 + e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)\left(c^2\left(-1 + e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right) + 8\,e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\pi^2\nu^2\sigma^2\right) - 8\,e^{\frac{8\pi^2\nu^2\sigma^2}{c^2}}\pi^2\nu^2\sigma^2\,\text{Erf}\left[\frac{2\pi\nu\sigma}{c}\right]^2\right)\Big/\left(8\left(-1 + e^{\frac{4\pi^2\nu^2\sigma^2}{c^2}}\right)^2\pi^2\sigma^2\right)$$

Namely, $(\Delta f)^2 = \frac{\nu^2((1-e^{-\gamma})((1-e^{-\gamma})+2\gamma)-2\gamma\,\text{Erf}[\gamma^{1/2}])}{2\gamma(1-e^{-\gamma})^2}$ which simplifies to $(\Delta f)^2 = \nu^2\,\frac{1}{2\gamma} = \frac{c^2}{8\pi^2\,\delta^2}$ when $\gamma \gg 1$. Therefore, $(\Delta x)^2\,(\Delta f)^2 = \frac{c^2}{16\pi^2}$.

Once interpreted as a photon, $(\Delta f)^2 = c^2 (\Delta p)^2 / h^2$, and hence $(\Delta x)^2\,(\Delta p)^2 = \frac{\hbar^2}{4}$, as expected.