# 129A Lecture Notes
## Strong Interactions II

# 1 Regge Trajectories and String

One organizing principle came out when people looked at the masses and the spins of hadrons of the same type (*i.e.*, same isospin, same parity, etc). By plotting the masses and spins on the so-called Chew–Frautschi plot[1] on the $(m^2, J)$ plane, the hadrons of the same type fall on straight lines: $J = \alpha(0) + \alpha' m^2$. The intercept $\alpha(0)$ depends on the types, but $\alpha'$ came out more-or-less the same: $\alpha' \simeq (1.2\text{–}1.4 \text{ GeV}^{-2})$.

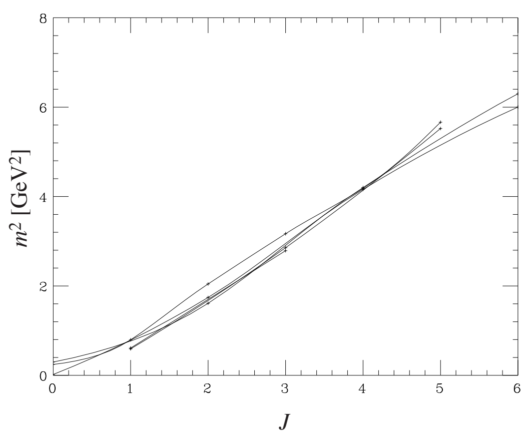


Figure 1: Chew–Frautschi plot for mesons.

This fact led to the following picture: the hadrons are elastic strings, not particles. When the string is stretched, there is a constant tension $T$, giving the energy $Tr$ where the $r$ is the length of the string. This was the beginning of the string theory. If you regard $Tr$ as a potential energy, a hand-waving analysis indeed gives the linear relation between $E^2 = (mc^2)^2$ and $J$. Write down the relativistic kinetic energy $cp$ and the linear potential $Tr$:

$$H = cp + Tr. \tag{1}$$

[1]This is Jeff Chew of our Department.

In the spirit of Bohr's argument, $pr = l\hbar$. Therefore,

$$E = \frac{\hbar c l}{r} + Tr. \tag{2}$$

Minimizing it with respect to $r$, we find the average length of the string $r = \sqrt{\hbar c l / T}$ and its energy

$$E = \frac{\hbar c l}{\sqrt{\hbar c l / T}} + T\sqrt{\hbar c l / T} = 2\sqrt{\hbar c l T}, \tag{3}$$

and hence

$$m^2 = (E/c^2)^2 = \frac{4T}{c} l. \tag{4}$$

Indeed, the mass-squared is linear with the angular momentum $l$![2]

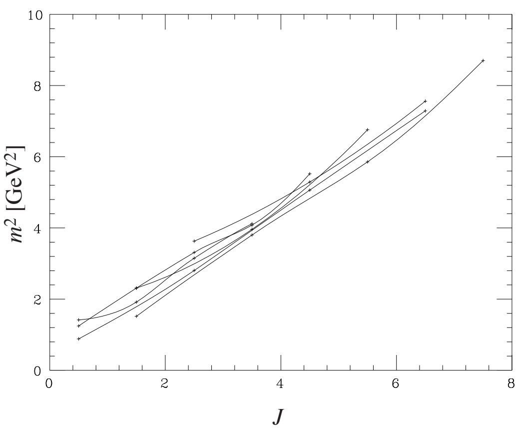


Figure 2: Chew–Frautschi plot for baryons.

[2]The correct quantum mechanical treatment of a relativistic string turned out to be much more difficult. You start with Nambu–Goto action and quantize it, and find that the quantization procedure is consistent only in 26-dimensional spacetime. Even that case predicts a tachyon, a particle with a negative mass-squared, whose presence violates causality because it would go faster than the speed of light. A supersymmetric version happily gets away with tachyons, but still live in 10-dimensional spacetime. But the interesting thing about it was that it predicts a massless spin-two particle, which we don't see in the world of hadrons but can be identified with the graviton. Since then the string theory switched its gear from the would-be theory of hadrons to the "Theory of Everything," including quantum gravity.

Even though this picture of elastic strings seems to reproduce the Chew–Frautschi relation qualitatively, what it meant was a (sort-of) return of Thomson-model of atoms: a jelly-like, fluffy, elastic, continuous medium rather than a hard-centered composite objects.

This picture was proven to be false experimentally by an experiment at SLAC, bombarding protons by very energetic (in the standard of their time) electrons. This experiment is a repetition of the Rutherford experiment, called Deep Inelastic Scattering experiment, except that his $\alpha$-particle is replaced by the electron (this is a good idea because the electron is truly elementary as far as we know and we don't need to worry about its structure to interpret the data) and the atom by the proton. Similarly to the Rutherford experiment, they saw electrons nearly backscattered: something impossible by an elastic string. What it means is that there are something hard and tiny inside the proton, which Feynman later called "partons." They indeed measured the form factor of the partons by studying the dependence of the cross section as a function of the momentum transfer, very similar to the discussion we had with the nuclear form factor. It turns out that the form factor is nearly independent of the momentum transfer, which implies that the "partons" are point-like, apparently behaving as free particles. (Remember the form factor is the Fourier transform of the charge distribution. A constant is the Fourier transform of the delta function.) See Chapter 8 of Cahn–Goldhaber for more details.

Even that didn't convince people that the proton was a composite object for a while. One of the main reason was that, in order to reproduce the observed pattern of hadrons in terms of point-like constituents, the "partons" had to have fractional electric charges, as pointed out by Gell-Mann and Néeman. Gell-Mann named them "quarks." The constituent of the nucleons and pions are supposed to be "up" and "down" quarks, with electric charges $+\frac{2}{3}|e|$ and $-\frac{1}{3}|e|$. Nobody (except a few wrong experiments) could find fractionally charged objects. We will see more details of this quark model in the next section.

# 2 Flavor $SU(3)$ and Quarks

Another organizing principle came when hadrons of similar masses are compared among each other. The first remarkable observation by Gell-Mann and

Nishijima was that all hadrons satisfy the following relationship,

$$Q = I_z + \frac{B+S}{2}. \tag{5}$$

Here, $Q$ is the electric charge, $I_z$ the 3rd component of the isospin, $B$ the baryon number, and $S$ the strangeness. You can verify this relation for all particles in Tables 1, 2, 3, 4. I have no idea how they came up with such a mysterious relation by looking at a big mess like this.

| particles | isospin | strangeness | quark content |
|---|---|---|---|
| $\pi^+$, $\pi^0$, $\pi^-$ | 1 | 0 | $u\bar{d}$, $(u\bar{u} - d\bar{d})/\sqrt{2}$, $d\bar{u}$ |
| $K^+$, $K^0$ | 1/2 | +1 | $u\bar{s}$, $d\bar{s}$ |
| $\overline{K^0}$, $K^-$ | 1/2 | −1 | $s\bar{d}$, $s\bar{u}$ |
| $\eta$ | 0 | 0 | $(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$ |
| $\eta'$ | 0 | 0 | $(u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$ |

Table 1: Members of pseudoscalar $(0^-)$ meson nonet. The quark contents shown for $\eta$ and $\eta'$ is the idealized octet and singlet combinations, but they actually mix substantially in reality.

| particles | isospin | strangeness | quark content |
|---|---|---|---|
| $\rho^+$, $\rho^0$, $\rho^-$ | 1 | 0 | $u\bar{d}$, $(u\bar{u} - d\bar{d})/\sqrt{2}$, $d\bar{u}$ |
| $K^{*+}$, $K^{*0}$ | 1/2 | +1 | $u\bar{s}$, $d\bar{s}$ |
| $\overline{K^{*0}}$, $K^{*-}$ | 1/2 | −1 | $s\bar{d}$, $s\bar{u}$ |
| $\omega$ | 0 | 0 | $(u\bar{u} + d\bar{d})/\sqrt{2}$ |
| $\phi$ | 0 | 0 | $s\bar{s}$ |

Table 2: Members of vector $(1^-)$ meson nonet. $\omega$ and $\phi$ are an "ideal mixing" of singlet and octet states such that $\phi$ is an $s\bar{s}$ to a good approximation.

Gell-Mann and Zweig generalized the concept of isospin to include strangeness. What they noticed is that the baryons come in an octet, which is a natural combination in the group $SU(3)$. In order not to get into too much complication, I discuss the flavor $SU(3)$ "top down," *i.e.*, going backwards in history knowing what the true theory turned out to be. The hadrons are composite objects built up of "quarks," spin 1/2 fermions that are never observed in isolation.[3] The mesons are bound states of one quark and one anti-quark,

[3]Apparently Gell-Mann took this name from Finnegans Wake by James Joyce, but I haven't read the book.

| particles | isospin | strangeness | quark content |
|---|---|---|---|
| $p$, $n$ | 1/2 | 0 | $uud$, $udd$ |
| $\Lambda^0$ | 0 | -1 | $uds$ |
| $\Sigma^+$, $\Sigma^0$, $\Sigma^-$ | 1 | -1 | $uus$, $uds$, $dds$ |
| $\Xi^0$, $\Xi^-$ | 1/2 | -2 | $uss$, $dss$ |

Table 3: Members of baryon octet of spin 1/2. They all have baryon number one.

| particles | isospin | strangeness | quark content |
|---|---|---|---|
| $\Delta^+$+, $\Delta^+$, $\Delta^0$, $\Delta^-$ | 3/2 | 0 | $uuu$, $uud$, $udd$, $ddd$ |
| $\Sigma^{*+}$, $\Sigma^{*0}$, $\Sigma^{*-}$ | 1 | -1 | $uus$, $uds$, $dds$ |
| $\Xi^{*0}$, $\Xi^{*-}$ | 1/2 | -2 | $uss$, $dss$ |
| $\Omega^-$ | 0 | -3 | $sss$ |

Table 4: Members of baryon decuplet of spin 3/2. They all have baryon number one.

and baryons of three quarks. The three quarks, up, down, and strange are listed in Table 5.

| name | notation | isospin | strangeness | electric charge |
|---|---|---|---|---|
| up-quark | $u$ | 1/2 | 0 | $+2/3$ |
| down-quark | $d$ | 1/2 | 0 | $-1/3$ |
| strange-quark | $s$ | 0 | $-1$ | $-1/3$ |

Table 5: Quantum numbers of quarks. All of them carry baryon number 1/3.

The minute you postulate that all hadrons are made up of quarks, Gell-Mann–Nishijima relation all of a sudden becomes trivial. It is simply that three quarks satisfy this relation. Note that $Q$, $I_z$, $B$, and $S$ are all *additive* quantum numbers, and Gell-Mann–Nishijima relation is a linear equation. Therefore, if the constituents (quarks) satisfy this relation, the bound states also satisfy the same relation. Nothing mysterious anymore.

Within the quark model, the isospin invariance of the strong interactions can be understood as a consequence of an approximate degeneracy between up and down quarks. If their masses are not very different, to the extent you ignore the electromagnetism, you can freely interchange or rotate among up and down quarks. A general rotation among up and down quarks is a two-by-two unitary matrix,

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow U \begin{pmatrix} u \\ d \end{pmatrix}. \tag{6}$$

But the overall phase change common to both up and down quarks is not very interesting; an overall phase does not do anything in quantum mechanics. Therefore, the "interesting" part is the set of two-by-two unitarity matrices without an overall phase factor, namely with unit determinant. These matrices form a "group," in the sense of group theory in mathematics. The definition of a group is very simple. (1) Any product of two elements of the group must also belong to the group. The multiplication is associative. (2) There must be an identity element. (3) There must be an inverse element. That's all. In the case of unitarity matrices, all of them are trivially satisfied, even with the requirement of unit determinant because the determinant of a product is a product of determinants.

This group of unit-determinant two-by-two unitarity matrices is called $SU(2)$, where $S$ stands for "special" meaning unit determinant, $U$ for unitarity, and 2 for two-by-two. The most general such matrix is

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1. \tag{7}$$

$a$, $b$ are complex numbers, and therefore there are four real parameters, subject to the constraint $|a|^2 + |b|^2 = 1$, leaving three real parameters for the group $SU(2)$. Given this most general form, it can be rewritten in terms of

$$U = I \cos|\vec{\theta}| + i \frac{\sin|\vec{\theta}|}{|\vec{\theta}|} (\vec{\tau} \cdot \vec{\theta}), \tag{8}$$

by identifying $a = \cos|\vec{\theta}| + i\frac{\sin|\vec{\theta}|}{|\vec{\theta}|}\theta_3$, $b = \frac{\sin|\vec{\theta}|}{|\vec{\theta}|}(\theta_2 + i\theta_1)$ which clearly satisfies the constraint. $\vec{\tau}$ are Pauli (Heisenberg) matrices for the isospin. Written in this form, one can further rewrite it as

$$U = \exp\left(i\vec{\tau} \cdot \vec{\theta}\right). \tag{9}$$

The exponential of a matrix is defined using the Taylor expansion. Because any element of the group can be written in terms of an exponential of a linear combination of matrices $\vec{\tau}$, and hence we say that Pauli matrices *generate* the group. Due to the analogy to the case of the real spin, we take the *generators* to be normally $\frac{1}{2}\vec{\tau}$. The generators are traceless, as a consequence of the unit-determinant condition of the matrices. This can be seen using the Cayley–Hamilton formula, $\det e^A = e^{\mathrm{Tr}A}$ for any matrix $A$.[4] There are four linearly independent two-by-two hermitian matrices, and indeed three are left after requiring tracelessness. The one that is left out is the two-by-two identity matrix.

Similarly, once we include the strange quark, the interchange among three quarks define a group $SU(3)$, the group of unit-determinant three-by-three unitarity matrices. The strong interaction is invariant under this, except for the "small" difference in masses. It is actually not that small (they differ by about 150 MeV$/c^2$), but it can still be considered a small perturbation to the hadron mass, say baryon masses of 940 MeV$/c^2$ and above. Under the $SU(3)$, three *flavors* of quarks transform as a triplet by definition. There are nine linearly independent three-by-three hermitian matrices, and eight of them are traceless. They are given by so-called Gell-Mann's lambda matrices (the generalization of Pauli's sigma matrices),

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{11}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{12}$$

All matrices are normalized as $\mathrm{Tr}\lambda^a\lambda^b = 2\delta^{ab}$, and we normally take $\frac{1}{2}\lambda^a$ as generators of the $SU(3)$ group.

Using the lambda matrices, the octet of mesons can be easily constructed. They are given by the combinations $\bar{q}\lambda^a q$ for $a = 1, \cdots, 8$ with

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad \bar{q} = (\bar{u}, \bar{d}, \bar{s}). \tag{13}$$

[4] If you want to prove this formula, the easiest way is to first diagonalize the matrix $A = PDP^{-1}$, and then compare both sides of the formula. It works even when $D$ is not completely diagonal but is of Jordan's standard form.

The quark vector change from $q \to Uq$ under the $SU(3)$, while the anti-quarks $\bar{q} \to \bar{q}U^\dagger$. This way, we can identify

$$\pi^+ = \bar{q}\frac{\lambda^1 - i\lambda^2}{2}q = \bar{u}d, \quad \pi^0 = \bar{q}\frac{\lambda^3}{\sqrt{2}}q = \frac{\bar{u}u - \bar{d}d}{\sqrt{2}}, \quad \pi^- = \bar{q}\frac{\lambda^1 + i\lambda^2}{2}q = \bar{d}u$$
$$K^+ = \bar{q}\frac{\lambda^4 - i\lambda^5}{2}q = \bar{s}u, \quad K^0 = \bar{q}\frac{\lambda^6 - i\lambda^7}{2}q = \bar{s}d$$
$$\overline{K^0} = \bar{q}\frac{\lambda^6 + i\lambda^7}{2}q = \bar{d}s, \quad K^- = \bar{q}\frac{\lambda^4 + i\lambda^5}{2}q = \bar{d}u, \quad \eta = \bar{q}\frac{\lambda^8}{\sqrt{2}}q = \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}}.$$

The eight meson states in the octet transform among each other under the $SU(3)$.

The singlet state in the nonet is a state that does not change under the $SU(3)$ group. There is a unique such combination,

$$\eta' = \frac{1}{\sqrt{3}}\bar{q}q = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s). \tag{14}$$

Now the interesting case is the baryons. Let us leave out the spin degrees of freedom for the moment. Because the quarks are fermions, their wave function must be totally anti-symmetric. Out from three quarks, we choose three quarks without allowing two of them are the same, and hence there is only one combination,

$$\frac{1}{\sqrt{6}}(uds + dsu + sud - usd - dus - sdu). \tag{15}$$

It does not appear that we can obtain decuplet or octet this way.

Let us for the moment say that quarks have totally symmetric wave function as if they are bosons. We will come back to the question why this is possible for quarks later. With this new rule, there are many combinations possible. We choose three out of three objects, allowing multiple uses of the same objects.[5] There are ${}_{3+3-1}C_3 = 10$ combinations. This nicely

[5] In general, if you choose $r$ out of $n$ objects allowing the multiple uses, the number of combinations is given by ${}_nH_r \equiv {}_{n+r-1}C_r$. This can be understood by preparing $r$ boxes and $n-1$ fences. You order boxes and fences in an arbitrary way. There are ${}_{n+r-1}C_r$ ways to do so. Once you picked a particular order, you fill the first object in the boxes until you meet the first fence, and then fill the second object in the boxes until you meet the second fence, and so on, until you see the $n-1$-th fence after which you fill the $n$-th object in the boxes. Because there are $r$ boxes all together, you have made a choice of $r$ out of $n$ objects allowing multiple uses.

matches the decuplet. The choices are indeed exactly those given in Table 4, except that every wave function is symmetried. For example, $\Sigma^{*0}$ is $(uds+dsu+sud+usd+dus+sdu)/\sqrt{6}$, and $\Delta^+$ is $(uud+udu+duu)/\sqrt{3}$. $\Delta^{++}$ has only one term, $uuu$. It seems to work.

Now we have to put the spin degrees of freedom in. The case of decuplet we've just seen corresponds to choosing the spin up state for all three quarks, naturally giving total spin of 3/2. That is indeed the spin of the decuplet baryons. By allowing both spin up and down states for each quarks, there are six objects to choose three quarks from. The number of combinations is then ${}_{6+3-1}C_3 = 56$, and 40 of them are the decuplets of $S_z = 3/2, 1/2, -1/2, -3/2$ spin states. The remaining 16 are the octets of $S_1 = 1/2, -1/2$. This way, we can construct all baryon wave functions uniquely.

For instance, let us see how we can construct the wave function of spin up proton. We first start with the simplest decuplet state,

$$|\Delta^{++}, S_z = 3/2\rangle = |u^\uparrow u^\uparrow u^\uparrow\rangle. \tag{16}$$

We first construct $\Delta^+$ by using the lowering operator in the isospin space,

$$|\Delta^+, S_z = 3/2\rangle = \frac{1}{\sqrt{3}} I^- |\Delta^{++}, S_z = 3/2\rangle = \frac{1}{\sqrt{3}} (|d^\uparrow u^\uparrow u^\uparrow\rangle + |u^\uparrow d^\uparrow u^\uparrow\rangle + |u^\uparrow u^\uparrow d^\uparrow\rangle). \tag{17}$$

The normalization factor $\sqrt{3}$ was obtained using the general formula (the same formula as in the case of the spin) $I^-|I, I_z\rangle = \sqrt{I(I+1) - I_z(I_z-1)}|I, I_z - 1\rangle$. Then we lower the spin to obtain $S_z = 1/2$ state of $\Delta^+$,

$$\begin{aligned} |\Delta^+, S_z = 1/2\rangle &= \frac{1}{\sqrt{3}} S^- |\Delta^+, S_z = 1/2\rangle \\ &= \frac{1}{3} S^- (|d^\uparrow u^\uparrow u^\uparrow\rangle + |u^\uparrow d^\uparrow u^\uparrow\rangle + |u^\uparrow u^\uparrow d^\uparrow\rangle) \\ &= \frac{1}{3} [(|d^\downarrow u^\uparrow u^\uparrow\rangle + |d^\uparrow u^\downarrow u^\uparrow\rangle + |d^\uparrow u^\uparrow u^\downarrow\rangle) + (|u^\downarrow d^\uparrow u^\uparrow\rangle + |u^\uparrow d^\downarrow u^\uparrow\rangle + |u^\uparrow d^\uparrow u^\downarrow\rangle) \\ &\quad + (|u^\downarrow u^\uparrow d^\uparrow\rangle + |u^\uparrow u^\downarrow d^\uparrow\rangle + |u^\uparrow u^\uparrow d^\downarrow\rangle)]. \end{aligned} \tag{18}$$

The job now is to find a totally symmetric combination of quarks with the same flavor orthogonal to this combination. We find

$$\begin{aligned} |p, S_z = 1/2\rangle = \frac{1}{\sqrt{18}} [&2|d^\downarrow u^\uparrow u^\uparrow\rangle + 2|u^\uparrow d^\downarrow u^\uparrow\rangle + 2|u^\uparrow u^\uparrow d^\downarrow\rangle \\ &- |d^\uparrow u^\downarrow u^\uparrow\rangle - |d^\uparrow u^\uparrow u^\downarrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - |u^\uparrow d^\uparrow u^\downarrow\rangle - |u^\downarrow u^\uparrow d^\uparrow\rangle - |u^\uparrow u^\downarrow d^\uparrow\rangle] \end{aligned} \tag{19}$$

Another way to construct the proton wave function is by combining one up and down quark in $I = 0$, $S = 0$ combination first,

$$|ud - du\rangle| \uparrow\downarrow - \downarrow\uparrow\rangle = |u^\uparrow d^\downarrow\rangle - |u^\downarrow d^\uparrow\rangle - |d^\uparrow u^\downarrow\rangle + |d^\downarrow u^\uparrow\rangle, \tag{20}$$

and then combine it with $u^\uparrow$ to form $I = 1/2$, $S = 1/2$ state,

$$|u^\uparrow d^\downarrow u^\uparrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - |d^\uparrow u^\downarrow u^\uparrow\rangle + |d^\downarrow u^\uparrow u^\uparrow\rangle. \tag{21}$$

Finally we make it totally symmetric by adding two cyclic permutations,

$$\begin{aligned} & |u^\uparrow d^\downarrow u^\uparrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - |d^\uparrow u^\downarrow u^\uparrow\rangle + |d^\downarrow u^\uparrow u^\uparrow\rangle \\ + & |d^\downarrow u^\uparrow u^\uparrow\rangle - |d^\uparrow u^\uparrow u^\downarrow\rangle - |u^\downarrow u^\uparrow d^\uparrow\rangle + |u^\uparrow u^\uparrow d^\downarrow\rangle \\ + & |u^\uparrow u^\uparrow d^\downarrow\rangle - |u^\uparrow u^\downarrow d^\uparrow\rangle - |u^\uparrow d^\uparrow u^\downarrow\rangle + |u^\uparrow d^\downarrow u^\uparrow\rangle. \end{aligned} \tag{22}$$

By reassembling them, we find exactly the same combination as before. All other baryon wave function can be constructed by similar methods.

See Griffiths how the $SU(3)$ wave functions can be used with the non-relativistic quark model to calculate the masses of mesons and baryons as well as baryon magnetic moments. The model works remarkably well.

# 3 Color $SU(3)$

The success of the non-relativistic quark model of baryons left a big question behind.

1. Why do we construct a totally *symmetric* wave function when the spin-statistics theorem tell us that quarks (having spin 1/2) must obey Fermi-Dirac statistics?

There are also several other important questions about the quark model.

2. Why don't we observe isolated quarks? Nobody has detected a fractional electric charge. If quarks are indeed constituents of the hadrons, somehow we have to come up with an explanation why quarks are "confined" inside hadrons.

3. What is the connection to the string-like behavior we saw in Regge trajectories?

4. Why do "partons" behave as if they are free at high-energy collisions (Deep Inelastic Scattering), if they are actually quarks that are confined by a very strong force inside hadrons?

The answer to these questions emerged from the most unlikely one about the statistics. Greenberg pointed out that if the quarks carry additional degrees of freedom that takes three possible states, and if the baryon is totally anti-symmetric for this degree of freedom, it will explain the totally symmetric wave function of the quarks. Later people called this degree of freedom "color," Red, Green, and Blue. They have nothing to do with the color we see optically with our eyes. The only connection is that there are three primary colors. Greenberg's assumption can be rephrased as the following: only "white" states are allowed in nature. In the case of baryons, three quarks come in different colors so that the color part of the wave function is $(\mathrm{RGB} + \mathrm{GBR} + \mathrm{BRG} - \mathrm{RBG} - \mathrm{GRB} - \mathrm{BGR})/\sqrt{6}$. It is "white" because three primary colors cancel each other, while the wave function is totally anti-symmetric. Then the complete baryon wave function is the produce of totally symmetric ones we constructed in the previous section and the totally anti-symmetric color wave function, and hence is totally anti-symmetric under the interchange of quarks as a whole. Then it is consistent with the Fermi-Dirac statistics of quarks. In the case of mesons, quarks can come in R, G, B, and anti-quarks in $\overline{\mathrm{R}}$, $\overline{\mathrm{G}}$, $\overline{\mathrm{B}}$. The meson wave function has then the color part, $(\mathrm{R}\overline{\mathrm{R}} + \mathrm{G}\overline{\mathrm{G}} + \mathrm{B}\overline{\mathrm{B}})/\sqrt{3}$. Again the wave function is "white" because the color is cancelled between quarks and anti-quarks.

The proposal of the color appears an even more bizzarre "patch" to the already-bizzarre quark model. But this turned out to be the true choice by the Nature.

We can freely rotate three colors among each other like we did with three flavors. This defines yet another $SU(3)$ group, the color $SU(3)$. Han and Nambu pointed out that maybe there is a color-octet vector (spin one) bosons coupled to the color degree of freedom, which explains why we see only color-neutral bound states in nature. These color-octet vector bosons are later called "gluons," reflecting the strong glue these vector bosons provide to confine quarks eternally inside hadrons.

Recall the QED. There, the photon couples to anything electrically charged, and the Feynman vertex is proportional to the charge, $e$ for the electron and $|e|$ for the proton, and $\frac{2}{3}|e|$ for the up quark, and so on. $e$ determines the strength of the electromagnetism in general, often quoted in terms of the fine-structure constant $\alpha = e^2/\hbar c = 1/137$ (at small momentum transfer) because it is a dimensionless combination. The coefficient in front of $|e|$, namely $-1$ for the electron, $+\frac{2}{3}$ for the up-quark, is specific to the particle. The exchange of the virtual photon produces the Coulomb potential, and it is responsible for binding the hydrogen atom together. In the same way, we have gluons that couple to anything colored. The Feynman vertex is given by three-by-three traceless hermitian matrices, namely the generators of $SU(3)$: $T^a = \lambda^a/2$. The Feynman vertex is given by $g_s T^a$, where $\alpha_s = g_s^2/4\pi\hbar c$ characterizes the strength of the gluon interaction, while $T^a$ is the analog of the particle-specific charge. The three-by-three matrix is specific to color-triplet object such as quarks. For anti-quarks, they are anti-triplets and the corresponding matrices are given by $-(T^a)^*$.

Because gluons couple to the $SU(3)$ generators $T^a$, there are eight gluons. For example, a "red" quark can become a "blue" quark, by emitting a gluon of color R$\overline{\text{B}}$. This corresponds to the combination $T^4 + iT^5$. The important difference from the QED is that the spin 1 force carrier itself is colored, while the photon is not charged.

This theory of quarks and gluons is called Quantum ChromoDynamics, or QCD.

# 4 QCD

It took a while before the QCD was accepted as the theory of strong interactions. The key discovery was that the QCD exhibits the property called "asymptotic freedom," found by 't Hooft, Politzer, and Gross–Wilczek in 1972.[6]

Recall the vacuum polarization effect in the QED. When you place an electrically charged object in a "vacuum," the vacuum "polarizes" because of the negative energy sea of electrons, or equivalently, of virtual pairs of electrons and positrons. This vacuum polarization effect makes the electric

[6]'t Hooft, the Nobel Laureate in 2000, actually didn't write a paper on it, but merely commented on it in a conference talk. Apparently he had not realized how important this was. Politzer was a graduate student at Harvard when he discovered it. There is a persistent rumor that Gross and his student Wilczek at Princeton, had a wrong sign, and corrected it only after they had a chance to listen to a seminar by Politzer. But Gross truly gets credit for realizing the importance of this particular sign and making it clear that this is the most important property that makes the QCD a serious candidate for the theory of strong interactions.

charge appear smaller at longer distances because of screening effect. The same effect happens in the QCD. The virtual pairs of quark-anti-quark screen the color you've put in the vacuum. However, there is an important difference. The Coulomb field is a virtual gluon, and because the gluon itself is colored, it also produces virtual pairs of gluons, which further produces gluons, and so on. The net effect of it is that gluons "anti-screen" color instead of screening it, and makes the color appear larger and larger at longer distances. In the case of the QED, we can parameterize the Coulomb force as $\frac{\alpha(r)\hbar c}{r}$, where $\alpha(r)$ is the distance-dependent fine-structure constant. At $r \to \infty$ limit, $\alpha(\infty) = 1/137$, while it "runs" to larger values for smaller distances. Similarly, we can parameterize the gluon-Coulomb force as $\frac{\alpha_s(r)\hbar c}{r}$, where $\alpha_s(r)$ is a "small" perturbative value at short distances and hence the force is approximately Coulombic. However at larger distances, $\alpha_s(r)$ keeps growing, and eventually the force approaches a constant value, the "tension" $T$. Once the force becomes constant, the potential energy is an integral of the force and hence grows linearly with distance. That is how quarks are confined inside hadrons.

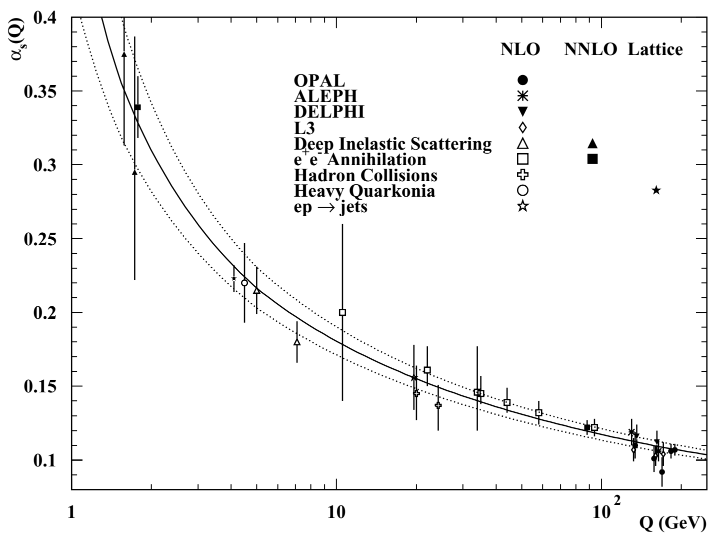


Figure 3: Experimentally measured values of $\alpha_s$ as a function of the energy scale. Taken from OPAL Collaboration, *Eur. Phys. J.* **C16**, 185 (2000).

If you try to pull a quark inside a meson, the gluon Coulomb force from the anti-quark becomes constant beyond the distance of about 0.5fm, and the potential energy between the quark and anti-quark keeps rising linearly. At some point, it becomes energetically favorable to create a pair of a quark and an anti-quark, so that the created quark binds with the original anti-quark to form a bound state meson, cutting off the linearly rising potential, and the created anti-quark binds with the original quark in the same way. This way, you never see an isolated quark. It is always in a bound state no matter what. This way, the QCD answers the second and third questions beautifully.

On the other hand, as you go to small distances, or large momentum transfers, the strong coupling constant becomes weak. For instance, at the momentum transfer of 91 GeV, $\alpha_s \approx 0.12$. This is a small enough value so that perturbation theory can be trusted. Then it is no mystery that the quarks can behave as if they are nearly free inside hadrons in Deep Inelastic Scattering experiments. This is the answer to the fourth question.

In the end, the theory of the strong interactions is based on the same principle as the QED. You fix the charge of the particle (electric charges for the QED and the multiplicity of color for the QCD), and the force carrier (spin one boson) couples according to the charge. Such theories are called gauge theories. The main difference is that gluons themselves carry color, while the photon does not have an electric charge. The latter type of theories is called Abelian gauge theories, while the former type the non-Abelian gauge theories. The important and immediate consequence of a gauge theory is that particles with the same charge should feel the same force. This point can be tested in the QCD by comparing the three-jet events for the bottom quark, charm quark, and other light quarks in electron-positron annihilation (see more on this later). The comparison is shown in Fig. 4.

Despite these successes, the prejudice against fractionally charged and enternally confined consituents you can never see was so strong, that people still didn't believe in quarks until November 11, 1974.

# 5 November Revolution

Only in the second half of 70's, after the so-called November Revolution in Particle Physics when the teams at SLAC and Brookhaven independently discovered a particle now called $J/\psi$, people started to take the quark model seriously. The $J/\psi$ is now understood as a boundstate of a charm quark and its anti-particle $c\bar{c}$, an entirely new type of quark not seen earlier. See

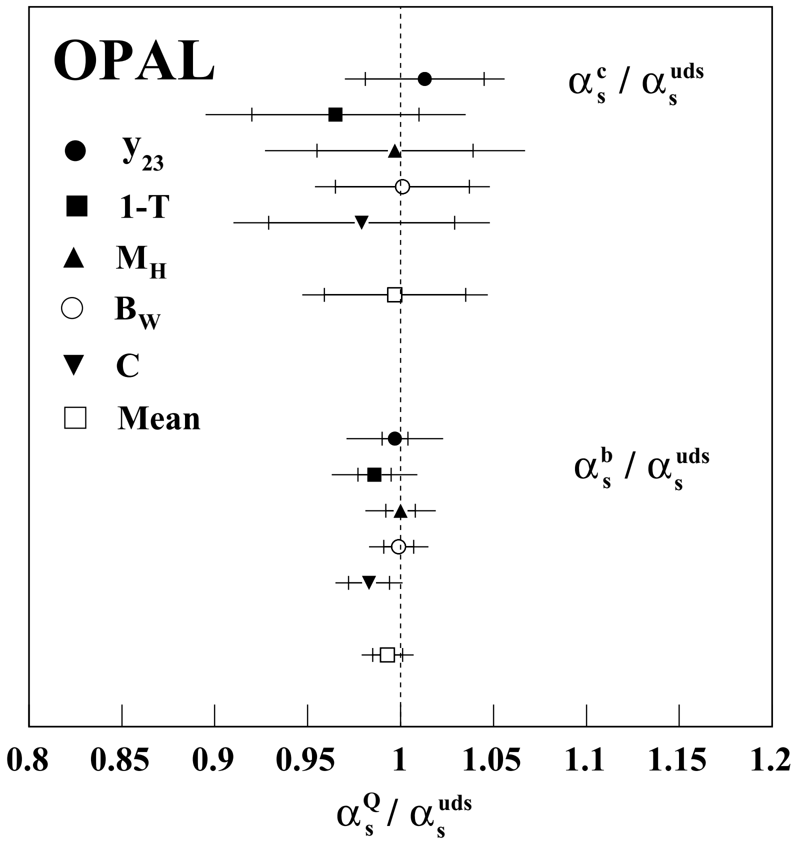


Figure 4: Comparison of the gluon coupling to different types of quarks using various methods. All of them are consistent with one, namely the universal strength to all quarks, supporting the gauge theory. Taken from OPAL Collaboration, *Eur. Phys. J.* **C16**, 407 (2000).

Chapter 9 of Cahn–Goldhaber for the fascinating story of the discoveries, including the Italian team who completely missed the Nobel prize by a few days. Afterwards, many more new mesons, and eventually a new baryon ($\Lambda_c(udc)$ baryon) was discovered, and everything fit together using the hypothesis of a new quark. People had to accept the quark hypothesis after a long period of skepticism.

See also Griffiths for the brief discussion of the spectroscopy of charmonium states, namely $c\bar{c}$ bound states. People tried to fit the observed spectrum of the charmonium using a non-relativistic potential between two massive particles, which confirmed the picture of the Coulombic potential at the short distance and the linear potential at the large distance, interpolated by an approximately logarithmic behavior at the intermediate distance. Griffiths gives an argument how the logarithmic behavior can be seen by comparing charmonium and later discovered bottomonium spectra.

# 6 Jets

Is there really no hope to "see" quarks and gluons? Actually there is. Because the quarks and gluons interact only weakly at high energies, you can create quarks and gluons in a perturbative process you can calculate, and "see" them after they "hadronize."

The best way to do so is using high-energy electron-positron colliders. We discussed before that the annihilation of an electron and a positron produces an energetic photon "at rest," a virtual photon with no momentum but with a large energy violating energy-momentum conservation. This virtual photon must materialize into something else quickly, but the photon doesn't remember that it was created by an electron and a positron. It can materialize into anything that has electric charge, as long as there is enough energy. For example, it can create a pair of muons $\mu^-\mu^+$. This way, you can produce particles that did not exist at all in the initial state.

In the same way, the virtual photon may materialize into a pair of quarks, whose Feynman vertex is proportional to the charge of the quark. There is an important difference, however, that the quarks come with three colors. Therefore the probability for prodeucing a quark pair is proportional to the electric charge squared, and multiplied by three for the possible final color

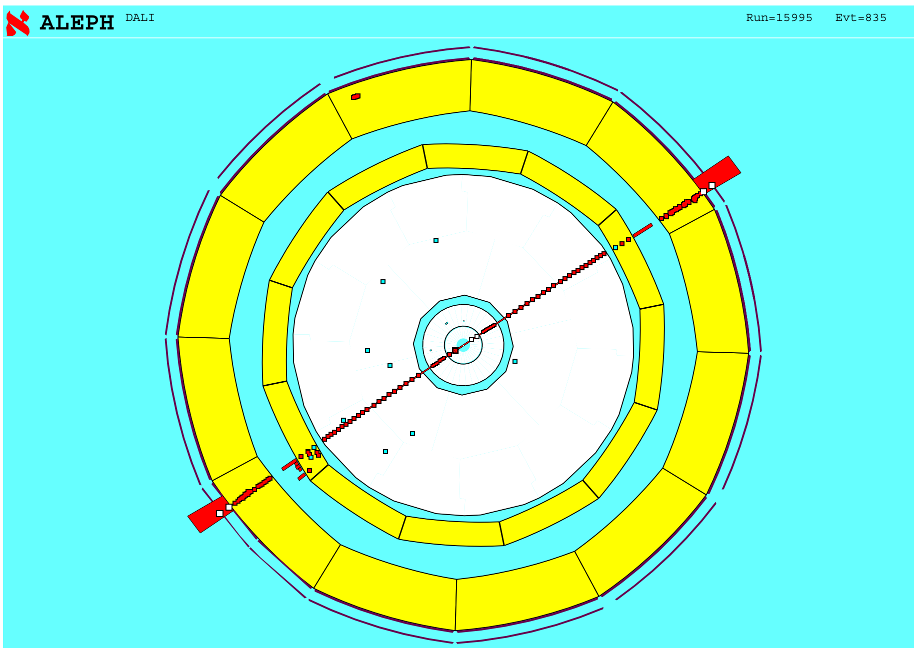


Figure 5: A pair of muons produced in electron-positron annihilation. The beams are perpendicular to the plane. Two charged tracks penetrate all layers and hit the outmost part of the detector called the muon chamber. From ALEPH DALI data base, `http://alephwww.cern.ch/DALI/`

states. In other words,

$$\frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3Q_q^2, \tag{23}$$

as long as your center-of-momentum energy is well above the threshold of producing the quark. Here, $Q_q$ is the charge of the quark in the unit of $|e|$. As long as the strong coupling constant is sufficiently weak, namely far above the threshold, this should be a good approximation to create quark pairs, and hence hadrons. Then the total production cross section of hadrons must be given by the so-called $R$-ratio,

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2. \tag{24}$$

The denominator is sometimes called the "point cross section," referring to the point-like nature of muons, and is given by

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{86.8 \text{ nb}}{(E_{CM}/\text{GeV})^2}. \tag{25}$$

One barn is 1 b $= 10^{-24}$ cm, and of course nanobarn is 1 nb $= 10^{-9}$ b. Below $J/\psi$, you add up-, down-, and strange-quarks and hence

$$R = 3\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2. \tag{26}$$

Above $J/\psi$, it is $\frac{10}{3}$. In fact, the rise in $R$ was one of the first indications for the existence of charm, which made Burton Richter and his team studying the hadronic cross section in small steps.

The most important message from this data is that the quark model prediction does not agree with the data by a large margin if there is no a factor of 3 coming from color degrees of freedom.

When the quarks are produced, they zoom out so energetically that the strong interaction is a small perturbation. However once the quark and anti-quarks are well separated, the confining force kicks in. They all of a sudden realize that they are supposed to be confined. The linearly rising potential is then minimized by producing new pairs of quarks and anti-quarks in between, and eventually many mesons (sometimes also baryons) are produced. In this way, the original quark and anti-quark become a collection of hadrons, still

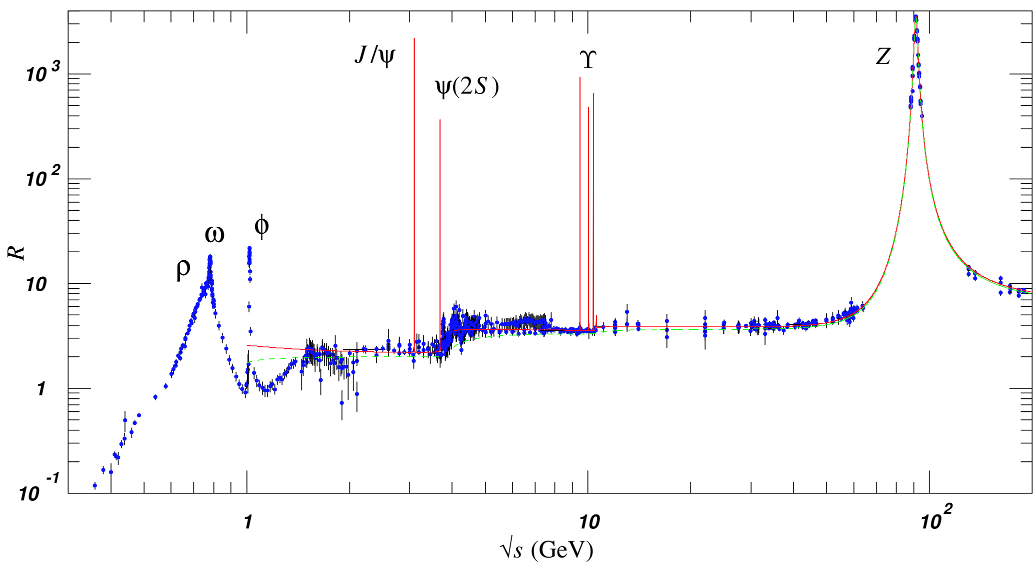


Figure 6: The data on $R$. The horizontal axis $\sqrt{s}$ is the center-of-momentum energy of the electron-positron collision. The resonances $J/\psi$, $\psi(2S)$, and $\Upsilon(nS)$ for $n = 1, \cdots, 4$ are also shown. The green dashed curve is the prediction of the native quark parton model Eq. (24), while the red solid curve is three-loop QCD prediction. Taken from `http://pdg.lbl.gov`.

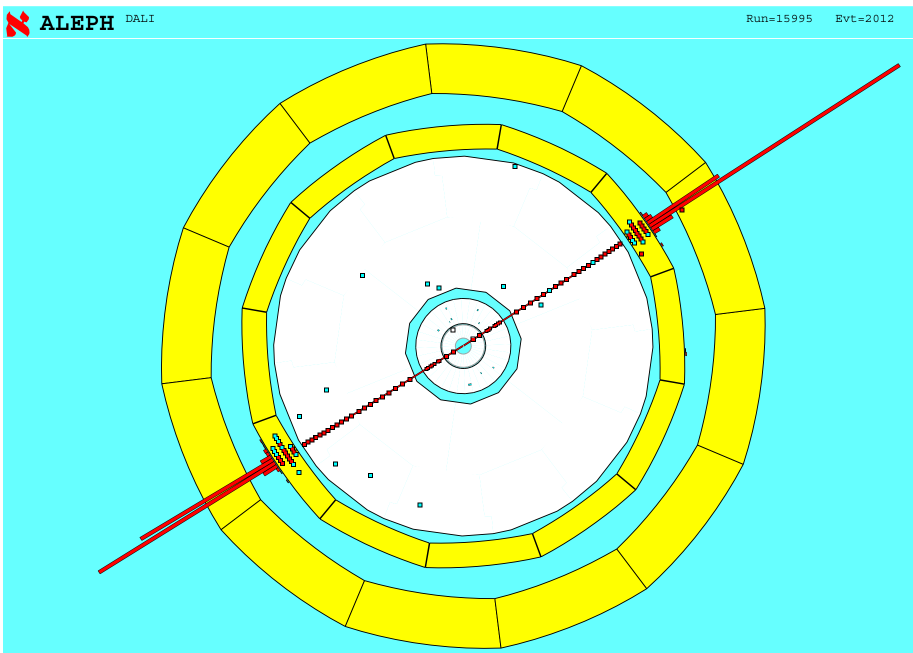


Figure 7: A pair of electron and positron produced in electron-positron annihilation. Two charged tracks shower in the electromagnetic calorimeters and stop.

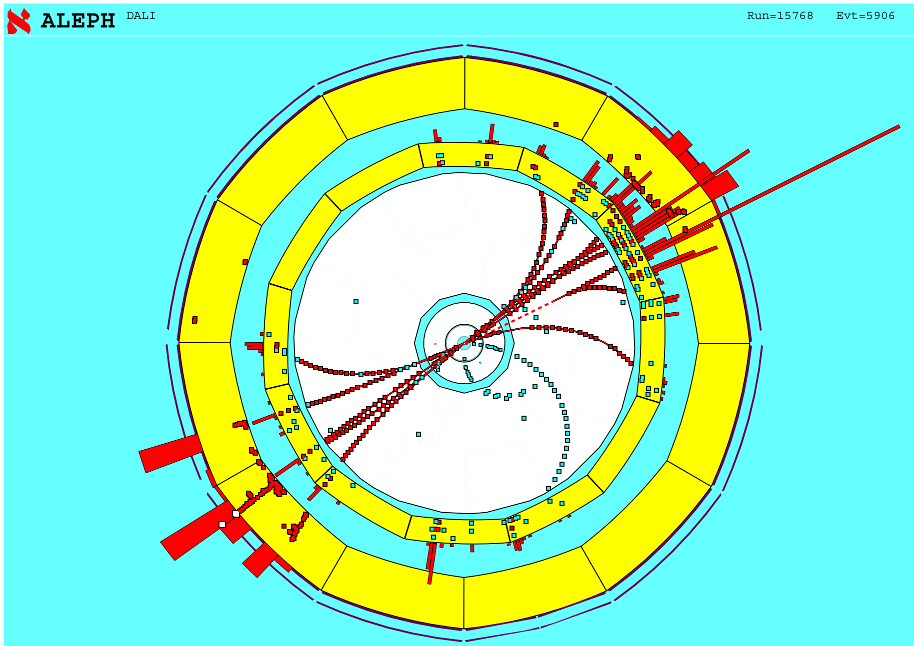


Figure 8: An event with two jets.

zooming out more-or-less along the original direction of the production. They form "jets."

By studying the properties of jets, we can infer properties of the quarks, hence we can "see" quarks. One of dramatic such examples is that we can determine the spin of quarks from the angular distribution of jets. When an electron and positron annihilate, they become a virtual photon and hence spin one. There is a selection rule that an electron with helicity $+1/2$ (*i.e.*, the spin aligned with its momentum) annihilates only a positron with helicity $-1/2$ and vice versa. In either case, the angular momentum along the beam axis is $\pm\hbar$. Similarly, the angular momentum along the produced quark direction is also $\pm\hbar$. For the combination of $+\hbar$ in the initial state along the beam direction and $+\hbar$ in the final state along the quark direction has the maximum overlap of angular momentum when the production angle $\theta = 0$, while the angular momentum would not be conserved if $\theta = \pi$. Therefore, the amplitude has the form $(1 + \cos\theta)$. For the $-\hbar$ case in the final state, the amplitude is proportional to $(1 - \cos\theta)$. The case of $-\hbar$ along the beam direction in the initial state can be discussed analogously. The angular distribution of the final state, summed over all possibilities, is then proportional to $(1+\cos\theta)^2 + (1-\cos\theta)^2 \propto 1+\cos^2\theta$. On the other hand, if the quarks had spin 0, they cannot be produced along the beam axis conserving angular momentum ($\pm\hbar$ vs 0). In this case, the amplitude would be proportional to $\sin\theta$ and hence the angular distribution $\sin^2\theta = 1-\cos^2\theta$. By studying the angular distribution of the jets, we should be able to distinguish these two possibilities. The data in Fig. 9 clearly shows that the spin of quarks is 1/2.

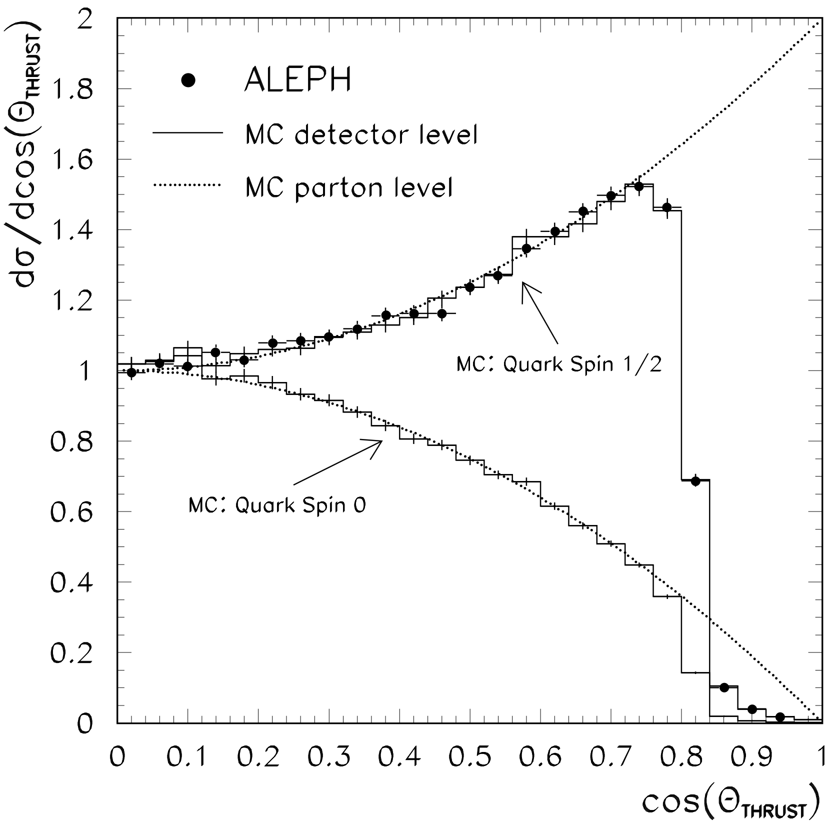


Figure 9: The angular distribution of two jet events is sensitive to the spin of the quark. The data clearly favor spin 1/2 quark. Taken from ALEPH Collaboration, *Phys. Rept.* **294**, 1 (1998).

When a gluon is emitted from either quark or anti-quark, you get three-jet events. Such events were first found at PETRA collider at DESY (Deutsches Electronen-SYnchrotron) in Hamburg, Germany, from late '70s to early '80s. This marked the "discovery" of gluons.

The angular correlations of three jets would be different depending on the

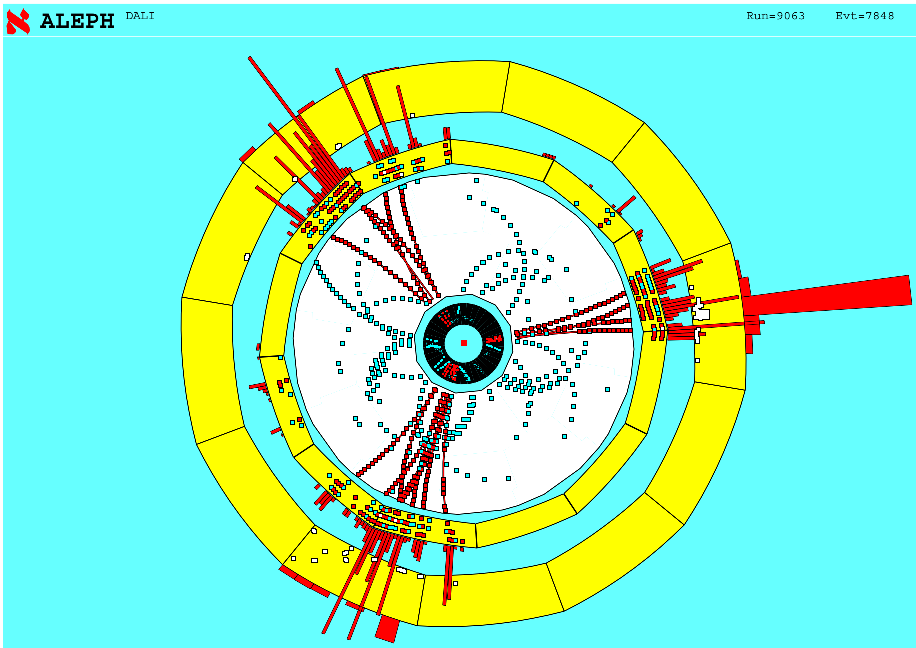


Figure 10: An event with three jets.

spin of the gluon. Fig. 11 shows that gluon spin is indeed one as expected.

Finally, there are several ways for four-jet events appear. One "boring" one is to radiate gluons twice from either quark or anti-quark, or one each. The second possibility is that either quark or anti-quark radiative a virtual gluon, which further materializes to a quark pair. This process measures how strongly a gluon couples to quarks, proportional to a parameter called $C_F$ defined by $C_F T^a T^a$. The third possibility is that the virtual gluon materializes to a pair of gluons instead. This process measures how strongly gluons couples among themselves, proportional to a parameter called $C_A$ defined by $C_A \delta^{ab} = f^{acd} f^{bcd}$ where the structure constant is defined by commutation relation among generators $[T^a, T^b] = i f^{abc} T^c$. For $SU(N)$, $C_F = (N^2 - 1)/2N$ and $C_A = N$, are different for other types of quarks. Another parameter $T_R = 1/2$ is basically the normalization of the generators, $\mathrm{Tr} T^a T^b = T_R \delta^{ab}$.

Experimentally these different mechanisms for four-jet events cannot be discriminated on event-by-event basis, but thanks to different angular distributions, they can be separated on statistical basis. This way, we determine these group theory factors experimentally. In particular, $C_F \neq 0$ tests that the gluons couple to themselves, proving the non-abelian nature. This was

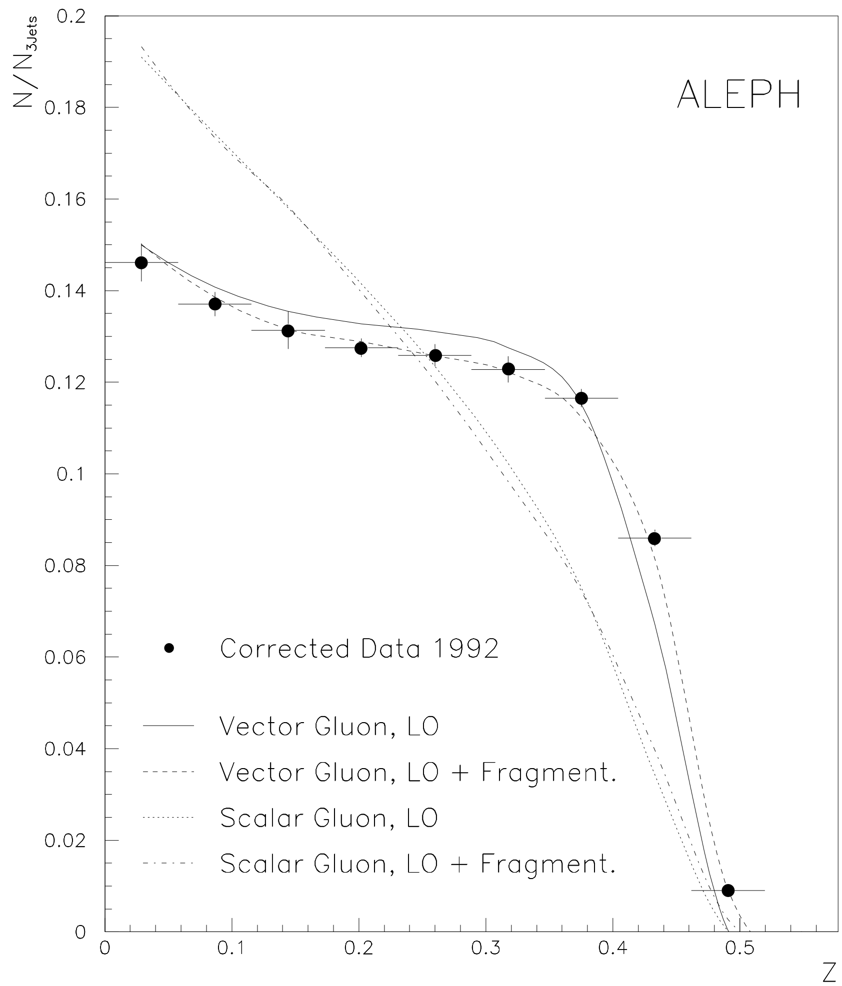


Figure 11: The angular distribution of three jet events is sensitive to the spin of the gluon. The data clearly favor spin 1 (vector) gluon over spin 0 (scalar) case. Taken from ALEPH Collaboration, *Phys. Rept.* **294**, 1 (1998).

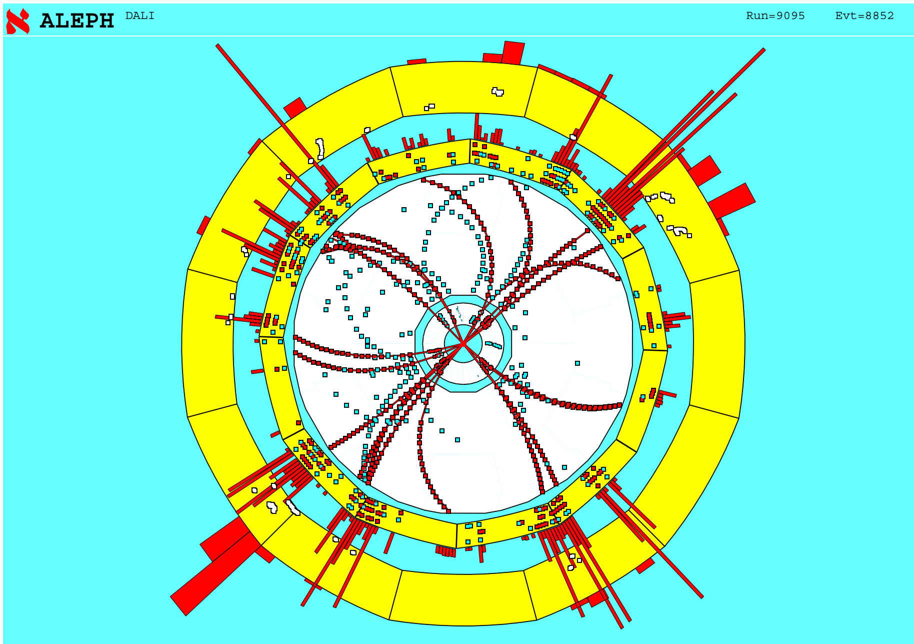


Figure 12: An event with four jets.

first observed at TRISTAN collider in Tsukuba, Japan.

# 7 Aftermath

What, after all, was the Yukawa's theory of nuclear force by the pion exchange, then? It is now understood as a van der Waals like force among bound states. The van der Waals force acts among neutral atoms, which do not have any overall electric charges. But the residual effects due to the quantum polarizability make them attract each other. In the case of nucleon-nucleon potential, the situation is similar but somewhat different. A proton is a bound state of *uud* quarks while a neutron of *udd* quarks. Because they share the same constituents, you may interchange them. If a proton wants to interchange one of its quarks with the neutron, it needs to "send", say, an up-quark to the neutron. But before the up-quark reaches the neutron, it starts feeling the linear potential, and realizes that it needs to be bound with something: it then creates a pair of, say, a down-quark and an anti-down quark. The created down-quark stays with the rest of the proton: it

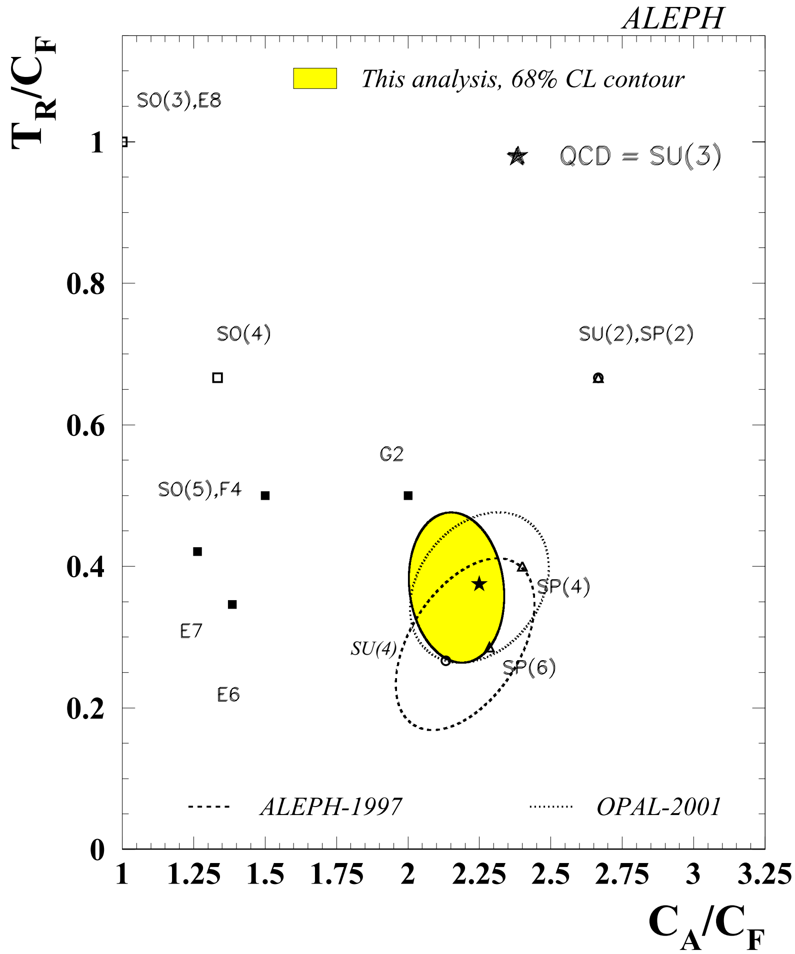


Figure 13: This analysis determined $C_A = 2.93 \pm 0.14 \pm 0.58$ and $C_F = 1.35 \pm 0.07 \pm 0.26$, compared to the $SU(3)$ prediction of 3 and 4/3. The plot shows the region preferred by data in yellow, with the $SU(3)$ prediction shown as a star. Taken from ALEPH Collaboration, CERN-EP-2002-029, `http://alice.cern.ch/format/showfull?sysnb=2309042`.

is now a $dud$ state and has become a neutron. The created anti-down quark $\bar{d}$ goes together with the "sent-out" up-quark forming a charged pion $u\bar{d}$. This charged pion can now propagate from what-used-to-be-a proton to the neutron. The $\bar{d}$ inside the charged pion annihilates together with one of the $d$ quarks in the neutron, and the up-quark gets together with the rest of the neutron. It is now a $uud$ state and has become a proton. This is the charge-exchange reaction via the one-pion exchange. It is still fine, except that it is only an effective description of what is truly going on useful only at relatively long distances (even though it is very short from the daily-life point of view).

It is ironic that the reason why the strong interaction is short-ranged is because it is truly long-ranged. It certainly sounds paradoxical but it is true. In the case of electromagnetism, we know the force is long-ranged. But in daily phenomena, we don't see ourselves getting attracted by the car on the other side of the street by the Coulomb force. The reason is that (almost) all matter has become neutral by the Coulomb attraction between nuclei and electrons. It does not leave any long-range force once they have become neutral, and the only truly long-range force one should worry about is gravity which cannot be neutralized (no anti-gravity!). In the same way, the strong force due to the gluon exchange is truly long-ranged, even more so than the electromagnetism. The force remains constant at large distances, while the Coulomb force does die away as the inverse square law. No matter can be colored because it is attracted to other colored matter, and they get confined. Once the color is neutralized (whitened), you do not see the strong force over long distances any more. The only residual force then is by exchanging something color-neutral, and the lightest among them was the pion. That is why the pion was important at a relatively long distance, but there was nothing really fundamental about the pion. It was simply the lightest hadron you could exchange.