Fall, 02

Physics 129A

Solutions to H/w #1

(1) Denote E = energy of the electron,

E' = energy of the positron,

$\mathcal{E}$ = energy of $\Upsilon(4S)$,

p, p', ρ = momenta of the electron, the positron, and $\Upsilon(4S)$ respectively

Conservation laws imply:

(1) $\mathcal{E} = E + E'$,

(2) $\rho = p - p'$ (p' is the absolute value of momentum, so we need a minus sign because the positron is going in the opposite direction of the electron)

We also know the masses of the three particles – denote them m for e^- and e^+ and M for $\Upsilon(4S)$

Energy and momentum of any particle are related through its mass, so we have:

(3), (4) $\frac{1}{c^2}E^2 - p^2 = m^2c^2 = \frac{1}{c^2}E'^2 - p'^2$

(5) $\frac{1}{c^2}\mathcal{E}^2 - \rho^2 = M^2c^2$

We have 5 equations with five unknowns: $E', \mathcal{E}, p, p', \rho$.

We need to solve for E':

$$(1),(2) \rightarrow (5) \quad \frac{1}{c^2}(E+E')^2 - (p-p')^2 = M^2c^2$$

$$\frac{1}{c^2}\left(E^2+2EE'+E'^2\right) - p^2 + 2pp' - p'^2 = M^2c^2$$

$$(3),(4) \longrightarrow \quad m^2c^2 + \frac{2EE'}{c^2} + m^2c^2 + 2pp' = M^2c^2$$

$$(\star) \qquad 2c^2pp' = \left(M^2c^4 - 2m^2c^4\right) - 2EE'$$

$$\left[A \equiv M^2c^4 - 2m^2c^4\right] \qquad 4c^4p^2p'^2 = A^2 - 4EE'A + 4E^2E'^2$$

$$(3),(4) \longrightarrow \quad 4\left(E^2 - m^2c^4\right)\left(E'^2 - m^2c^4\right) = A^2 - 4EE'A + 4E^2E'^2$$

$$0 = 4E'^2\left(E^2 - m^2c^4 - E^2\right) + 4EAE' - \left(A^2 + 4m^2c^4\left(E^2 - m^2c^4\right)\right)$$

$$0 = 4m^2c^4E'^2 - 4EAE' + \underbrace{\left(M^4c^8 + 4m^2c^4\left(E^2 - M^2c^4\right)\right)}_{B}$$

$$E' = \frac{4EA \pm \sqrt{16E^2A^2 - 16m^2c^4B}}{8m^2c^4}$$

$$= \frac{4Ec^4\left(M^2 - 2m^2\right) \pm \sqrt{16E^2c^8\left(M^2 - 2m^2\right)^2 - 16m^2c^8\left(M^4c^4 + 4m^2\left(E^2 - M^2c^4\right)\right)}}{8m^2c^4}$$

$$= E\,\frac{M^2 - 2m^2}{2m^2} \pm \frac{1}{2m^2}\sqrt{E^2\left(M^4 - 4M^2m^2\right) - m^2M^2c^4\left(M^2 - 4m^2\right)}$$

$$= E\,\frac{M^2 - 2m^2}{2m^2} \pm \frac{M}{2m^2}\sqrt{\left(M^2 - 4m^2\right)\left(E^2 - m^2c^4\right)}$$

We have: $M = 10.58 \frac{GeV}{c^2}$, $m = 0.511 \frac{MeV}{c^2}$ (from Griffiths)

So $\frac{M^2 - 2m^2}{2m^2} = \frac{1}{2}\left(\frac{M}{m}\right)^2 - 1 = \frac{1}{2}\left(\frac{10.58 \times 10^9}{0.511 \times 10^6}\right)^2 - 1 = 2.14 \times 10^8$

$$\frac{M}{2m^2} = \frac{10.58 \times 10^9}{2 \times (0.511 \times 10^6)^2} \left(\frac{eV}{c^2}\right)^{-1} = 2.03 \times 10^{-2} \frac{c^2}{eV}$$

$$M^2 - 4m^2 \simeq M^2 = 1.12 \times 10^{20} \frac{eV^2}{c^4}$$

For $E = 9.0\ GeV$, we get:

- $E \frac{M^2 - 2m^2}{2m^2} = 9.0\ GeV \times 2.14 \times 10^8 = 1.93 \times 10^{18}\ eV$

- $\frac{M}{2m^2}\sqrt{(M^2 - 4m^2)(E^2 - m^2c^4)} \approx \frac{M}{2m^2}\sqrt{M^2E^2} = \frac{M^2}{2m^2}E \simeq \frac{M^2 - 2m^2}{2m^2}E =$
$= 1.93 \times 10^{18}\ eV$

The precision of our calculations is not sufficient to account for the difference between the two quantities, but it turns out that the solution we need is actually the difference, not the sum. To see that that's true, first consider $E'_+ = 2 \times 1.93 \times 10^{18}\ eV \simeq \frac{M^2}{m^2}E$

It's easy to see that it's not a solution of (✱), though it is a solution of the equation that follows (✱) — we introduced an extra solution when we squared (✱).

Indeed, the RHS (right-hand side) of (*) is

$$RHS \approx M^2c^4 - 2E\frac{M^2}{m^2}E = \frac{M^2}{m^2}\left(m^2c^4 - 2E^2\right) < 0$$

whereas, of course,

$$LHS > 0$$

So LHS and RHS have different signs, but, when squared, produce the same result — that's how the extra solution appeared.

So what we need is E'_-. We could use a computer to calculate both quantities to a great precision and subtract them or we can be more clever and manage using only our calculator. We know that

$$m^2 \ll M^2, \quad m^2c^4 \ll E^2,$$

so we can expand the square root:

$$\sqrt{(M^2-4m^2)(E^2-m^2c^4)} = \sqrt{\underbrace{M^2E^2}_{\text{big}} - \underbrace{(4m^2E^2 + m^2c^4M^2)}_{\text{smaller}} + \underbrace{4m^4c^4}_{\text{very small}}}$$

$$= ME\sqrt{1 + \frac{-(4m^2E^2 + m^2c^4M^2) + 4m^4c^4}{M^2E^2}}$$

$$= ME\left(1 - \frac{1}{2}\,\frac{4m^2E^2 + m^2c^4M^2}{M^2E^2} + \text{smaller terms}\right)$$

Then

$$E'_- \approx E\,\frac{M^2-2m^2}{2m^2} - \frac{M}{2m^2}ME\left(1 - \frac{1}{2}\,\frac{+4m^2E^2+m^2c^4M^2}{M^2E^2}\right)$$

$$= E\frac{M^2}{2m^2} - E - \frac{M^2}{2m^2}E + E + \frac{M^2c^4}{4E} = \frac{M^2c^4}{4E}$$

So for $E = 9.0$ GeV, we get:

$$E' = \frac{(Mc^2)^2}{4E} = \frac{(10.58\times 10^9\,eV)^2}{4\times 9.0\times 10^9\,eV} = \boxed{3.11\times 10^9\,eV}$$ in SLAC

The exact same story repeats for $E = 8.0$ GeV, and so:

$$E' = \frac{(10.58\times 10^9\,eV)^2}{4\times 8.0\times 10^9\,eV} = \boxed{3.50\times 10^9\,eV}$$ in KEK

Note that our calculations would be greatly simplified if we noticed right away that $E \gg mc^2$ and $E' \gg mc^2$, so (3) and (4) give $E \simeq pc$, $E' \simeq p'c$. Plugging that into (2), we'd get $c\wp = E - E'$. Then, using (1),

$$M^2c^4 = \mathcal{E}^2 - c^2\wp^2 = (E+E')^2 - (E-E')^2 = 4EE'$$

and

$$\boxed{E' = \frac{M^2c^4}{4E}}$$

The lesson is: the art of approximation is a very useful thing in physics.

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(2.) Denote: E_1, E_2, p_1, p_2 – the energies and momenta of the B – mesons,

m_B – their mass

In the rest frame of $\Upsilon(4S)$, the conservation laws give:

$$\begin{cases} Mc^2 = E_1 + E_2 \\ 0 = p_1 - p_2 \end{cases}$$

$$p_1 = p_2 \Rightarrow E_1 = E_2 \Rightarrow E_1 = E_2 = \frac{Mc^2}{2} = \frac{10.58\,GeV}{2}$$

$$\boxed{E_1 = E_2 = 5.29\,GeV}$$

$$p_1 = p_2 = \frac{1}{c}\sqrt{E_1^2 - m_B^2c^4} = \frac{1}{c}\sqrt{(5.29\times10^9 eV)^2 - (5279\times10^6 eV)^2}$$

$$= \frac{3.41\times10^8 eV}{c}$$

$$\boxed{p_1 = p_2 = 0.34\,GeV/c}$$

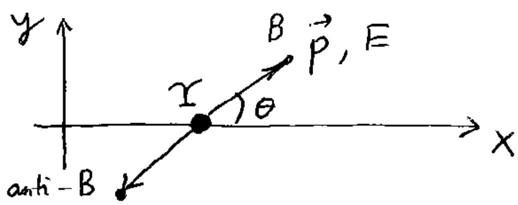
(3.) If a meson lives a time τ in its rest frame it will live $\gamma\tau$ in the Υ frame (usual time dilation). We also know that $p = \gamma m v$, $E = \gamma mc^2$, so it will travel

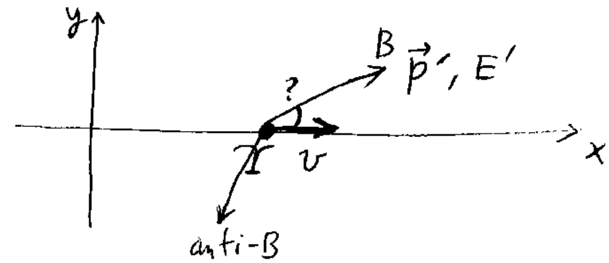
$$d = \gamma\tau v = \tau\gamma v = \tau\frac{p}{m_B} = 1.54\times10^{-12}s\ \frac{0.34\times10^9 eV}{3\times10^8 m/s}\ \frac{1\ c^2}{5279\times10^6 eV}$$

$$\boxed{d = 2.97\times10^{-5} m}$$

(4) Denote : γ - the dilation factor for the moving $\Upsilon(4S)$

K - rest frame of $\Upsilon(4S)$: K′ - lab frame:





p – the momentum of B in the $\Upsilon(4S)$ frame

p′ – the momentum of B in the lab frame, etc.

So in the K - frame,

$$\vec{p} = (p\cos\theta,\ p\sin\theta,\ 0)$$

and the energy-momentum four-vector for B is $\begin{pmatrix} E/c \\ p\cos\theta \\ p\sin\theta \\ 0 \end{pmatrix}$

Therefore in the K′ - frame the energy-momentum four-vector for B will be (by the usual transformation law)

$$\begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta & 0 & 0 \\ +\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p\cos\theta \\ p\sin\theta \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\gamma E}{c} + \gamma\beta\, p\cos\theta \\ +\gamma\beta\frac{E}{c} + \gamma\, p\cos\theta \\ p\sin\theta \\ 0 \end{pmatrix}}$$

+, not –, because K′ is moving to the left w/r to K

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(5) In problem 3, we found an easy formula for the distance traveled. In the lab frame, a meson will live $\tilde{\gamma}\tau$, so it will travel a distance $d = \tilde{\gamma}\tilde{v}\tau = \tilde{\gamma} m_B \tilde{v}\frac{\tau}{m_B} = \tilde{p}\frac{\tau}{m_B}$, where $\tilde{p}$ – the momentum of the meson in the lab frame.

In the previous problem, we found the expression for the four-momentum of the B-meson as a function of θ, so

$$\tilde{p} = \sqrt{\left(+\gamma\beta\frac{E}{c} + \gamma p\cos\theta\right)^2 + p^2\sin^2\theta} \quad ,$$

where p, E – the momentum and energy of the B-meson in the frame of $\Upsilon(4S)$ was calculated in problem 2 and γ, β – the dilation factors for the moving $\Upsilon(4S)$ can be obtained from the results of problem 1 after a few additional calculations.

We have for the average momentum:

$$\langle\tilde{p}\rangle = \int_{\cos\theta=-1}^{\cos\theta=1} \tilde{p}(\theta)\,dP(\theta) = \int_{-1}^{1} \sqrt{\left(+\gamma\beta\frac{E}{c} + \gamma p\cos\theta\right)^2 + p^2(1-\cos^2\theta)}\;\frac{3}{8}\left(1+\cos^2\theta\right)d\cos\theta$$

$$= \frac{3}{8}\int_{-1}^{1} \sqrt{\left(+\gamma\beta\frac{E}{c} + \gamma p x\right)^2 + p^2(1-x^2)}\;\left(1+x^2\right)dx$$

It is possible to take this integral analytically (integrals of the form $\int R\left(\sqrt{ax^2+bx+c}\,, x\right)dx$, where R is any rational function of two variables, can be taken, for example, by completing the square, getting it into one of the ~~two~~ three forms: $R(x, \sqrt{a^2-x^2})$, $R(x, \sqrt{x^2-a^2})$, $R(x, \sqrt{x^2+a^2})$, using the substitutions $x = a\sin t$, $x = a\cosh t$, $x = a\sinh t$ respectively, then using the substitutions $y = \tan\frac{t}{2}$, $y = \tanh\frac{t}{2}$, $y = \tanh\frac{t}{2}$ respectively), but it is more convenient to use a graphing calculator or a computer program. In problem 2, we found that

$$p = 0.34 \frac{GeV}{c}, \quad E = 5.29\ GeV$$

In problem 1, we found the energy of the positron for the two values of the energy of the electron. The total energy of $\Upsilon(4S)$ is the sum of the energies of the electron and positron, so

for SLAC, $\mathcal{E} = 9.0\,GeV + 3.11\,GeV = 12.11\,GeV$

$$\gamma = \frac{\mathcal{E}}{Mc^2} = \frac{12.11\ GeV}{10.58\ GeV} = 1.14$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \sqrt{1-\frac{1}{\gamma^2}} = 0.49$$

$$\gamma\beta \frac{E}{c} = 2.95 \frac{GeV}{c},$$

$$\gamma p = 0.39 \frac{GeV}{c}$$

$$p^2 = 0.12 \left(\frac{GeV}{c}\right)^2$$

for KEK, $\mathcal{E} = 8.0\,GeV + 3.5\,GeV = 11.5\,GeV$, $\gamma = \frac{11.5}{10.58} = 1.09$

$$\gamma\beta = \sqrt{\gamma^2 - 1} = 4.26, \quad \gamma\beta\frac{E}{c} = 2.25 \frac{GeV}{c}$$

$$\gamma p = 0.37 \frac{GeV}{c}, \quad p^2 = 0.12 \left(\frac{GeV}{c}\right)^2$$

We plug these values into Mathematica to find:

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for SLAC, $\langle \tilde{p} \rangle = 2.96 \frac{\text{GeV}}{c}$

for KEK, $\langle \tilde{p} \rangle = 2.27 \frac{\text{GeV}}{c}$

Then the average distance the meson will travel in the lab is

$$\langle d \rangle = \langle \tilde{p} \rangle \frac{\tau}{m_B} = \begin{cases} 2.96 \frac{\text{GeV}}{c} \frac{1.54 \times 10^{-12}\,\text{s}}{5279\,\text{MeV}/c^2} = \boxed{2.59 \times 10^{-4}\,\text{m}, \text{ SLAC}} \\ 2.27 \frac{\text{GeV}}{c} \frac{1.54 \times 10^{-12}\,\text{s}}{5279\,\text{MeV}/c^2} = \boxed{1.99 \times 10^{-4}\,\text{m}, \text{ KEK}} \end{cases}$$